21-270 **Introduction to Mathematical Finance**

Homework #13: Due Wednesday, April 15. **Spring 2020**

1. Consider a one-period binomial model with interest rate $r > 0$, initial stock price $S_0 > 0$ and up- and down-factors given by

$$u = 1 + r + \sigma$$
$$d = 1 + r - \sigma$$

where $0 < \sigma < 1 + r$.

In this model, let C denote a call option on the stock with expiration date 1, and strike price K satisfying $dS_0 < K < uS_0$. Let P denote a put option on the stock with the same expiration date and strike price.

(a) Find the risk neutral probabilities $\{\tilde{p}, \tilde{q}\}$ for this model.

(b) Let $\rho : \{H, T\} \to \mathbb{R}$ denote the return on the stock. Verify by direct computation that the risk-neutral expected value of ρ is $\widetilde{\mathbb{E}}[\rho] = r$.

(c) Show that the variance with respect to the risk-neutral probability measure, $\mathrm{Var}^{\widetilde{\mathbb{P}}}(\rho)$ is σ^2.

The parameter σ is the standard deviation of the return of the stock, often referred to as the *volatility* of the stock.

(d) Show that the arbitrage-free price of the call option C is an increasing function of the volatility σ.

(e) Show that the arbitrage-free price of the put option P is also an increasing function of the volatility σ.

Holders of put and call options are sometimes said to be "long volatility" because these securities tend to increase in value when volatility increases. Conversely, holders of short positions in put and call options are "short volatility." Many investors use options to "trade volatility" (i.e. making investments based on their view of the future volatility of a stock) while hedging against changes in value of the underlying stock.

2. Consider a one-period binomial financial model, with a stock and a bank. The bank offers the one period interest rate $r \geq 0$ for borrowing or depositing. The stock has initial price S_0. The price of the stock at time 1 is a random variable $S_1 : \{H, T\} \to \mathbb{R}$. This model has a risk-neutral probability measure $\widetilde{\mathbb{P}}$, with $\widetilde{\mathbb{P}}(H) = \tilde{p}$ and $\widetilde{\mathbb{P}}(T) = \tilde{q}$.

Consider a derivative security V whose payment at time 1 is a random variable $V_1 : \{H, T\} \to \mathbb{R}$. Let

$$\Delta = \frac{V_1(H) - V_1(T)}{S_1(H) - S_1(T)}$$

Show that a portfolio X that holds the security V and is short Δ shares of stock will have a constant value at time 1, i.e. $X_1(H) = X_1(T)$.

This method of reducing the volatility of a portfolio is sometimes referred to as "delta hedging."

3. [**Exercise 4.7**]
4. [**Exercise 4.11**]
5. [**Exercise 5.11**]
6. [**Exercise 5.19**]