Exercise 1 (Using duration to hedge against interest rate movements). In this exercise, we have two bonds. Each of these bonds pays a coupon once per year, has a nominal coupon rate $q[1] = 5\%$, and has a face value $F = 1000$. The first bond has one year left to maturity, that is, the first bond will pay $1050$ at time $t = 1$. The second bond has two years left to maturity, that is, the second bond will pay $50$ at $t = 1$ and $1050$ at $t = 2$. The prices at $t = 0$ of these bonds are

$$P_1 = 1007.39, \quad P_2 = 1002.35,$$

where the subscript denotes the number of years left to maturity.

(i) Show that $R_s(1) = 4.23\%$ and $R_s(2) = 4.89\%$.

(ii) Consider a portfolio that holds one of the one-year bonds and one of the two-year bonds. Show that the effective yield for this portfolio is $R_t = 4.66\%$. (In practice, the portfolio might hold 1000 of each of these bonds, and so the gains and losses described in the following parts of this discussion would be multiplied by 1000. We consider the case of one bond of each type in order to simplify the arithmetic.)

(iii) Show that the duration of the portfolio in part (ii) is $D = 1.48$.

(iv) In the event of a “parallel shift” in interest rates, which is a change in all the spot rates by the same amount, the value of the portfolio of bonds in (ii) will respond in approximately the same way as a portfolio of $N$ zero-coupon bonds, all of which have the same maturity $D = 1.48$, and where $N$ is chosen so that the $N$ zero-coupon bonds have the same value before the interest rate shift as the portfolio in part (ii). Since $D$ is close to 1.5, we shall consider zero-coupon bonds with maturity 1.5. We are not told the spot rate $R_s(1.5)$, but it is reasonable to assume it is close
to the average of the spot rates $R_s(1)$ and $R_s(2)$, which were computed in part (i). Therefore, we assume

$$R_s(1.5) = \frac{R_s(1) + R_s(2)}{2} = 4.56\%.$$  

Under this assumption about $R_s(1.5)$, determine the price $Z$ at $t = 0$ of a zero-coupon bond with maturity 1.5 and face value $F = \$100$. To the nearest whole number, how many of these bonds would it take to have the same value at $t = 0$ as the portfolio in part (ii)? Call this number $N$

(v) An investor holds the portfolio of bonds described in part (ii), but worries that at the next meeting of the Federal Reserve Board, the interest rate will be increased and this will cause her portfolio to lose value. Therefore, she shorts $N$ zero-coupon bonds with maturity 1.5 just before the meeting, which takes place at time $t = 0$. We assume that all the prices we have computed above are valid just before the meeting. Suppose that as a result of the meeting, all the interest rates increase by 50 basis points (1/2 of a percent) to new values

$$R_s^{\text{new}}(1) = 4.73\%, \quad R_s^{\text{new}}(1.5) = 5.06\%, \quad R_s^{\text{new}}(2) = 5.39\%.$$  

How much does the investor’s portfolio that holds the one-year and the two-year bond lose in value?

(vi) Under the assumptions of part (v), how much does the investor’s short position in the 1.5-year zero-coupon bond increase in value.

(v) Under the assumptions of part (v), how much does the investor’s portfolio of three bonds (the one-year coupon bond, the 1.5-year zero-coupon bond, and the two-year coupon bond) change in value?

**Exercise 2 (Exercise 28 from Chapter 2).** (Proof of Proposition 2.41) The purpose of this exercise is to lead you through a proof of Proposition 2.41. The proof is straightforward, but the algebra is a bit tricky.

(a) Show that for each positive integer $k$ and for every real number $\lambda \neq 1$ we have

$$\sum_{i=1}^{k} i\lambda^i = \left(\frac{\lambda}{1 - \lambda}\right)^2 \left(\frac{1}{\lambda} - \lambda^k\right) - (k + 1) \left(\frac{\lambda}{1 - \lambda}\right) \lambda^k$$

(59)
(Suggestion: Differentiate (16) with respect \( \lambda \) and multiply the result by \( \lambda \).

(b) Let

\[
\lambda = \frac{1}{(1 + R_I)^{1/m}} = \frac{1}{1 + \frac{r_I[m]}{m}},
\]

and notice that

\[
\frac{\lambda}{1 - \lambda} = \frac{m}{r_I[m]}.
\]

Use (16) and (59) with \( k = mn \) and use appropriate choices for \( T_i \) and \( F_i \) to evaluate the right-hand side of (47) and obtain the formula

\[
D = \frac{nr_I[m] + q[m]}{r_I[m] + q[m]} \left( \frac{1}{r_I[m]} \left[ \left( 1 + \frac{r_I[m]}{m} \right)^{mn+1} - 1 \right] - n - \frac{1}{m} \right).
\]  
(60)

(It will be helpful to divide the numerator and denominator by \( \lambda^{mn} \).)

(c) Show that the numerator on the right-hand side of (60) can be rewritten as

\[
\left( \frac{1}{r_I[m]} + \frac{1}{m} \right) \left[ r_I[m] + q[m] \left( 1 + \frac{r_I[m]}{m} \right)^{mn} - q[m] \right] - \frac{r_I[m]}{m} - nq[m] + nr_I[m],
\]

and use this fact to obtain (49) from (60).

Exercise 3 (Exercise 32 of Chapter 2). The 10-year effective spot rate is \( R_e(10) = 4.87\% \) and annuities that have maturity 10 years and make payments of $100 four times per year are trading at $3,187.31. Find the swap rate \( q^{swap}[4] \)
for an interest rate swap having maturity 10 years and 4 swap dates per year. Warning: The face value $F$ appearing in formula (58) in the lecture notes should not be there. The formula for the swap rate is

$$q^{\text{swap}}[m] = \frac{m(1 - D(n))}{\sum_{i=1}^{mn} D(\frac{i}{m})}.$$ 

*Exercise 4 (to be submitted). Let $n$ be a positive integer. Let $m$ be another positive integer, and for $i = 1, 2, \ldots, mn$, set $T_i = \frac{i}{m}$. Finally let $k$ be an integer between 0 and $mn$.

A forward starting swap is like a swap, except that it begins at time $t = T_k$ rather than at time $t = 0$. In other words, payments are made at times $T_{k+1}, T_{k+2}, \ldots, T_{mn}$. If $k = 0$, this is just the swap described in Section 2.11.2 of the notes. In particular, for the forward starting swap, one party, whom we call party $A$, pays the amount

$$F \frac{p_{i-1}[m]}{m}$$

to party $B$ at times $T_i$, $i = k + 1, k + 2, \ldots, mn$. Party $B$ pays to party $A$ the amount

$$F \frac{q[m]}{m}$$

at times $T_i$, $i = k + 1, k + 2, \ldots, mn$. What should the value of $q[m]$ be so that this contract has value 0 at time 0. (This value of $q[m]$ is called the forward swap rate.)

**Exercise 5.** Consider a zero-coupon bond with face value $F$ and maturity $T_b$. Let $T_d$ satisfy $0 \leq T_d \leq T_b$. Consider a forward contract that is entered at time $t = 0$ with delivery date $T_d$ on the zero-coupon bond. Let $F$ denote the forward price associated with this contract. An investor holding the long forward position must pay $F$ at time $T_d$ and receives $F$ at time $T_b$.

(i) What effective interest rate has the investor who takes the long forward position locked in at $t = 0$ for the investment of $F$ over the interval of time between $T_d$ and $T_b$?
(ii) What is the relationship between your answer in part (i) and the forward interest rate $R_{0,T_d,T_b}^{for}$?

**Exercise 6.** In 1461 King Edward IV of England borrowed the equivalent of $384 from New College of Oxford. He promptly repaid $160, but never repaid the remaining $224. In 1996, an administrator at New College discovered a record of this debt and contacted the queen asking for repayment of the original $224 together with interest compounded annually for 535 years.

1. The administrator originally suggested an annual rate of $R = 4\%$. (Here we are writing $R$ because with annual compounding there is no distinction between nominal and effective rates.) Calculate the value of the debt after 535 years using this interest rate.

2. After the queen refused to pay the amount calculated in part (a), the administrator suggested using the rate $R = 2\%$ instead, indicating that this greatly reduced amount would be enough to help the college with needed renovations. Calculate the value of the debt using annual compounding at this rate for 535 years. The queen refused to pay this amount as well, and to the best of our knowledge, the debt still remains unpaid.

*Exercise 7 (Exercise 2 of Chapter 3) – to be submitted.* Consider a coupon bond issued today with face value $F = \$1,000$ and maturity 10 years. The bond pays coupons twice per year at the nominal rate $q[2] = 5\%$. The current price of the bond is $P = \$1,000$. Given that $R_s(.5) = 4.2\%$ and $R_s(1) = 4.4\%$ find the arbitrage-free forward price $F_{0,1}$ for delivery of the bond at time 1, just after the coupon payment has been made. (The holder of the long position on the forward contract will receive her first coupon payment 6 months after delivery of the bond.)

**Exercise 8 (Exercise 4 of Chapter 3).** The current exchange rate for the Canadian Dollar is $0.6298$. In other words, to buy 100 Canadian Dollars we need to pay $62.98. The effective spot rate for maturity 3 months in the United States is $R_s^U(.25) = 3\%$ and the effective spot rate for maturity 3 months in Canada is $R_s^C(.25) = 2\%$. The price of a European put option on 100,000 Canadian Dollars with exercise date $T = .25$ and strike price $\$64,120$ ($0.6412$
per Canadian Dollar) is $1,100 ($0.011 per Canadian Dollar). The put option gives its holder the right to sell 100,000 Canadian Dollars at the strike price at time $T = .25$.

Compute the arbitrage-free price of a European call option on the same amount of Canadian Dollars and with the same exercise date and strike price.