21-270 **Introduction to Mathematical Finance**

Homework #11: Due Wednesday, April 1 **Spring 2020**

1. [**Exercise 3.14**] Assume that there is an ideal money market with constant effective interest rate $R > 0$. Let $T > 0$ and $K > 0$ be given. Consider a stock S that pays no dividends and let C^E and P^A denote a European call and an American put on the stock, both having expiration date T and strike price K. Show that
$$P_0^A + S_0 \leq C_0^E + K.$$
(Suggestion: Show that, no matter when the put option is exercised, an investor who purchases one European call and invests K in the bank at $t = 0$ is always at least as well off as an investor who purchases an American put and one share of stock at $t = 0$.)

2. [**Exercise 3.15**] Assume that there is an ideal money market with constant effective rate $R = 0$ and let S be a stock that does not pay dividends. Let $T > 0$ and $K > 0$ be given and let P^A and P^E be American and European put options on S both having expiration date T and strike price K. Show that
$$P_0^A = P_0^E.$$

3. [**Exercise 3.24**] Let $T > 0$, $K > 0$, and $\alpha \in (0, 1)$ be given and let S be a stock. Consider a (European-stye) derivative security V that pays its holder the amount
$$V_T = \max\{\alpha S_T, S_T - K\}$$
at time T. Determine constants β, γ, ad $\hat{K}$ such that the arbitrage-free price V_0 of this security is given by
$$V_0 = \beta S_0 + \gamma \hat{C}_0,$$
where $\hat{C}$ is a European call option on S with expiration T and strike price $\hat{K}$. (The constants that you find may depend on α and K.)

4. Consider two different probability spaces, both having the sample space $\Omega = \{\omega_1, \omega_2, \omega_3\}$ but with different probability measures $\mathbb{P}$ and $\hat{\mathbb{P}}$ given by
$$\mathbb{P}(\omega_1) = \frac{1}{3}, \quad \mathbb{P}(\omega_2) = \frac{1}{3}, \quad \mathbb{P}(\omega_3) = \frac{1}{3}$$
and
$$\hat{\mathbb{P}}(\omega_1) = \frac{1}{3}, \quad \hat{\mathbb{P}}(\omega_2) = \frac{2}{9}, \quad \hat{\mathbb{P}}(\omega_3) = \frac{4}{9.}$$
Let X be a random variable on Ω defined by
$$X(\omega_1) = 5, \quad X(\omega_2) = 1, \quad X(\omega_3) = -2$$
Determine each of the following:

(a) $\mathbb{P}[X \geq 0]$

(b) $\hat{\mathbb{P}}[X \geq 0]$

(c) $\mathbb{E}^{\mathbb{P}}[X]$

(d) $\mathbb{E}^{\hat{\mathbb{P}}}[X]$

(e) $\mathrm{Var}^{\mathbb{P}}(X)$

(f) $\mathrm{Var}^{\hat{\mathbb{P}}}(X)$

5. **[Exercise 4.2]** Let $(\Omega, \mathbb{P})$ be a finite probability space and let $A, B \subset \Omega$ be events. Using the definition of the probability $\mathbb{P}[U]$ of an event $U \subset \Omega$ show that

   (a) $\mathbb{P}[A \cup B] = \mathbb{P}[A] + \mathbb{P}[B] - \mathbb{P}[A \cap B]$.

   (b) if $A \subset B$ then $\mathbb{P}[A] \leq \mathbb{P}[B]$.

   (c) $\mathbb{P}[A \cap B] \geq \mathbb{P}[A] + \mathbb{P}[B] - 1$.