FUNDAMENTAL SETS OF SOLUTIONS AND THE
WRONSKIAN

2006 SPRING

Goal: Given a linear system of homogeneous, constant coefficient, first order differential equations
\[
\frac{dx}{dt} = Ax
\]
we would like to find a general solution, i.e. an expression that allows, by choosing various parameters, a solution satisfying any initial condition
\[
x(t_0) = w.
\]
Assuming that \( A \) is an \( n \times n \) matrix, our intention is to find a set of \( n \) solutions \( \{x_1(t), \ldots, x_n(t)\} \) so that the linear combination
\[
x_g(t) = c_1x_1(t) + \cdots + c_nx_n(t)
\]
is the general solution. In this case \( \{x_1(t), \ldots, x_n(t)\} \) is called a fundamental set of solutions.

1. BACKGROUND

The procedure we shall follow depends on some results from linear algebra, which we will state here without proof.

Definition 1. A basis for \( \mathbb{R}^n \) is a set of vectors \( \{v_1, \ldots, v_k\} \) such that for any \( w \in \mathbb{R}^n \), there is a unique representation
\[
w = c_1v_1 + \cdots + c_kv_k.
\]

Theorem 1. If \( \{v_1, \ldots, v_k\} \) is a basis for \( \mathbb{R}^n \), then \( k = n \).

Example: The set
\[
\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}
\]
is a basis for \( \mathbb{R}^3 \).

Another important result we need from linear algebra is

Theorem 2. A set of \( n \)-vectors, \( \{v_1, \ldots, v_n\} \), is a basis for \( \mathbb{R}^n \) if and only if \( \{v_1, \ldots, v_n\} \) is a linearly independent set.
2. The Wronskian

With the linear algebra background above, the requirements for a fundamental set of solutions can now be restated as

**Definition 2.** The solutions \{x_1(t), \ldots, x_n(t)\} to the system of differential equations \( \frac{dx}{dt} = Ax \) form a fundamental set if for each \( t_0 \in \mathbb{R} \),

\[ \{x_1(t_0), \ldots, x_n(t_0)\} \]

is a basis for \( \mathbb{R}^n \).

By Theorem 2, \{x_1(t), \ldots, x_n(t)\} is a fundamental set if and only if:

\[ c_1x_1(t) + \cdots + c_nx_n(t) = 0 \quad \Rightarrow \quad c_1 = \cdots = c_n = 0. \]

Using matrix notation, this condition can be restated as:

\[
\begin{bmatrix}
x_1(t) & \cdots & x_n(t)
\end{bmatrix}
\begin{bmatrix}
c_1 \\
\vdots \\
c_n
\end{bmatrix} = 0 \quad \Rightarrow \quad
\begin{bmatrix}
c_1 \\
\vdots \\
c_n
\end{bmatrix} = 0
\]

This is equivalent to the statement

\[
\begin{bmatrix}
x_1(t) & \cdots & x_n(t)
\end{bmatrix}
\text{ is invertible.}
\]

Finally, this can be restated in terms of the determinant

\[
\det \begin{bmatrix}
x_1(t) & \cdots & x_n(t)
\end{bmatrix} \neq 0.
\]

Since the Wronskian is simply the determinant on the left:

\[
W[x_1, \ldots, x_n](t) = \det \begin{bmatrix}
x_1(t) & \cdots & x_n(t)
\end{bmatrix}
\]

we reach the conclusion that:

The solutions \{x_1(t), \ldots, x_n(t)\} to the system of differential equations \( \frac{dx}{dt} = Ax \) form a fundamental set if and only if the wronskian \( W[x_1, \ldots, x_n](t) \) is non-zero for all values of \( t \).