FUNDAMENTAL SETS OF SOLUTIONS AND THE WRONSKIAN

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Goal: Given a linear system of homogeneous, constant coefficient, first order differential equations

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x}$$

we would like to find a *general solution*, i.e. an expression that allows, by choosing various parameters, a solution satisfying any *initial condition*

$$\mathbf{x}(t_0) = \mathbf{w}.$$

Assuming that A is an $n \times n$ matrix, our intention is to find a set of n solutions $\{\mathbf{x}_1(t), \ldots, \mathbf{x}_n(t)\}$ so that the linear combination

$$\mathbf{x}_g(t) = c_1\mathbf{x}_1(t) + \cdots + c_n\mathbf{x}_n(t)$$

is the general solution. In this case $\{\mathbf{x}_1(t), \ldots, \mathbf{x}_n(t)\}$ is called a *fundamental set* of solutions.

1. Background

The procedure we shall follow depends on some results from linear algebra, which we will state here without proof.

Definition 1. A *basis* for $\mathbb{R}^n$ is a set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ such that for any $\mathbf{w} \in \mathbb{R}^n$, there is a unique representation

$$\mathbf{w} = c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k.$$

Theorem 1. *If $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is a basis for $\mathbb{R}^n$, then $k = n$.*

Example: The set

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

is a basis for $\mathbb{R}^3$.

Another important result we need from linear algebra is

Theorem 2. *A set of n-vectors, $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$, is a basis for $\mathbb{R}^n$ if and only if $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a linearly indemendent set.*

2. The Wronskian

With the linear algebra background above, the requirements for a fundamental set of solutions can now be restated as

Definition 2. The solutions $\{\mathbf{x}_1(t), \dots, \mathbf{x}_n(t)\}$ to the system of differential equations $\frac{d\mathbf{x}}{dt} = A\mathbf{x}$ form a fundamental set if for each $t_0 \in \mathbb{R}$,

$$\{\mathbf{x}_1(t_0), \dots, \mathbf{x}_n(t_0)\}$$

is a basis for $\mathbb{R}^n$.

By Theorem 2, $\{\mathbf{x}_1(t), \dots, \mathbf{x}_n(t)\}$ is a fundamental set if and only if:

$$c_1\mathbf{x}_1(t) + \cdots + c_n\mathbf{x}_n(t) = \mathbf{0} \quad \Longrightarrow \quad c_1 = \cdots = c_n = 0.$$

Using matrix notation, this condition can be restated as:

$$\begin{bmatrix} \mathbf{x}_1(t) & \dots & \mathbf{x}_n(t) \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = \mathbf{0} \quad \Longrightarrow \quad \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = \mathbf{0}$$

This is equivalent to the statement

$$\begin{bmatrix} \mathbf{x}_1(t) & \dots & \mathbf{x}_n(t) \end{bmatrix} \quad \text{is invertible.}$$

Finally, this can be restated in terms of the determinant

$$\det \begin{bmatrix} \mathbf{x}_1(t) & \dots & \mathbf{x}_n(t) \end{bmatrix} \neq 0.$$

Since the Wronskian is simply the determinant on the left:

$$W[\mathbf{x}_1, \dots, \mathbf{x}_n](t) = \det \begin{bmatrix} \mathbf{x}_1(t) & \dots & \mathbf{x}_n(t) \end{bmatrix}$$

we reach the conclusion that:

The solutions $\{\mathbf{x}_1(t), \dots, \mathbf{x}_n(t)\}$ to the system of differential equations $\frac{d\mathbf{x}}{dt} = A\mathbf{x}$ form a fundamental set if and only if the wrondkian $W[\mathbf{x}_1, \dots, \mathbf{x}_n](t)$ is non-zero for all values of t.