Exam #3 Review

1. Consider the system of differential equations

$$\begin{aligned}\frac{dx}{dt} &= -5x + 2y \\ \frac{dy}{dt} &= -x - 2y\end{aligned}$$

   (a) Find the solution of the system that satisfies the initial conditions $x(0) = 2$, $y(0) = 3$.

   (b) Sketch graph that shows the trajectory of the solution you found in part (a). On a second set of axes, sketch a phase portrait for the system of differential equations.

2. Consider the second order, linear differential equation

$$y'' + y = \mathcal{U}(t - \pi) + k\delta(t - \frac{3\pi}{2}).$$

   which models the motion of a mass-spring system under the influence of an external force.

   (a) Find the solution to the equation satisfying the initial conditions $y(0) = 0, y'(0) = 0$ using the Laplace transform method. It may help to know that $\frac{1}{s(s^2+1)} = \frac{1}{s} - \frac{s}{s^2+1}$.

   (b) Is there a value of k that results in the mass remaining stationary for $t > \frac{3\pi}{2}$?

3. Consider the system of differential equations

$$\begin{aligned}x' &= x + 2y \\ y' &= -5x - y\end{aligned}$$

   (a) Find the solution to the system satisfying the initial conditions $x(0) = -2$, $y(0) = 1$.

   (b) Find the x- and y-nullclines for the system. Use the nullclines and information from the analytic solution you found in (a) to sketch a phase portrait for the system.

   (c) Make a mark next to each term that describes this system:

__ center	__ spiral	__ sink	__ node	__ saddle
__ stable	__ asymptotically stable	__ source	__ simplex	__ unstable

4. Consider the differential equation

$$x'' + 4x = 2\delta(t - \pi) + \delta(t - T_1) + \delta(t - T_2)$$

   where $\pi < T_1 < T_2$.

(a) Find a solution to this equation satisfying $x(0) = 0$, $x'(0) = 0$. Your answer should depend on T_1 and T_2.

(b) Are there values for T_1 and T_2 such that $x(t) = 0$ for all $t > T_2$?

5. Consider the system of equations

$$\begin{aligned} \frac{dx}{dt} &= \quad\quad\; y \\ \frac{dx}{dt} &= 2x + y \end{aligned}$$

(a) Find the solution of the system satisfying the initial condition $x(0) = 3$, $y(0) = 2$.

(b) Describe the behavior of this solution (in the phase plane) as $t \to \infty$.

(c) Describe the behavior of this solution (in the phase plane) as $t \to -\infty$.

(d) Sketch a graph of this solution.

(e) Make a mark next to each term that describes this system:

__ center	__ spiral	__ sink	__ node	__ saddle
__ stable	__ asymptotically stable	__ source	__ simplex	__ unstable

6. Consider the system of equations

$$\begin{aligned} \frac{dx}{dt} &= 7x - y \\ \frac{dx}{dt} &= x + 5y \end{aligned}$$

(a) Find a value of A such that $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = e^{6t} \begin{bmatrix} 4 \\ 5 \end{bmatrix} + te^{6t} \begin{bmatrix} A \\ A \end{bmatrix}$ is a solution to the system.

(b) Determine the nullclines for this system. Where in the phase plane do solutions move upward? Downward? To the left? To the right? Sketch a graph showing the null clines, and the directions solutions travel when they cross the nullclines.

(c) Sketch a phase portrait for this system.

(d) Make a mark next to each term that describes this system:

__ center	__ spiral	__ sink	__ node	__ saddle
__ stable	__ asymptotically stable	__ source	__ simplex	__ unstable

7. Consider the system of differential equations

$$\begin{aligned} \frac{dx}{dt} &= -5x + 2y \\ \frac{dy}{dt} &= -x - 2y \end{aligned}$$

(a) Find the solution of the system that satisfies the initial conditions $x(0) = 2$, $y(0) = 3$.

(b) Sketch graph that shows the trajectory of the solution you found in part (a). On a second set of axes, sketch a phase portrait for the system of differential equations.

8. Consider the system of differential equations

$$\begin{aligned} \frac{dx}{dt} &= y - x^2 \\ \frac{dy}{dt} &= x - y^2 \end{aligned}$$

(a) Find the x-nullclines, where $\frac{dx}{dt} = 0$. Sketch the x-nullclines in the xy-plane. Determine where in the plane solutions are traveling to the right and where they are traveling to the left.

(b) Find the y-nullclines, where $\frac{dy}{dt} = 0$. Sketch the y-nullclines in the xy-plane. Determine where in the plane solutions are traveling upward and where they are traveling downward.

(c) Use the information from parts (a) and (b) to sketch a phase portrait for the system.

A Short List of Laplace Transforms

	Function	Transform
1.	1	$\frac{1}{s}$
2.	e^{at}	$\frac{1}{s-a}$
3.	t^n	$\frac{n!}{s^{n+1}}$
5.	$\sin(kt)$	$\frac{k}{s^2+k^2}$
6.	$\cos(kt)$	$\frac{s}{s^2+k^2}$
?.	$t\cos(kt)$	$\frac{s^2-k^2}{(s^2+k^2)^2}$
?.	$t\sin(kt)$	$\frac{2ks}{(s^2+k^2)^2}$
9.	$e^{at}\sin(kt)$	$\frac{k}{(s-a)^2+k^2}$
10.	$e^{at}\cos(kt)$	$\frac{s-a}{(s-a)^2+k^2}$
12.	$u_c(t)$	$\frac{e^{-cs}}{s}$
13.	$u_c(t)f(t-c)$	$e^{-cs}F(s)$
14.	$e^{ct}f(t)$	$F(s-c)$
17.	$\delta(t-c)$	e^{-cs}
18(a).	$f'(t)$	$sF(s)-f(0)$
18(b).	$f''(t)$	$s^2F(s)-sf(0)-f'(0)$