21-260 **Differential Equations** **D. Handron**

Exam #2 — Review Problems

Closed book and notes; calculators not permitted. Be sure to show all work and explain your reasoning as clearly as possible. Good Luck.

1. A system of vats and pipes is constructed in the following way:

   Vat 1 initially contains 100 ℓ of water and 25g of salt. Vat 2 initially contains 100 ℓ of water and 10g of salt. Vat 2 initially contains 50 ℓ of water and 50g of salt.

   Pure water is added to Vat 1 at a rate of 5ℓ per minute. Two spigots allow the solution to flow from Vat 1 to Vats 2 at a rate of 4ℓ per minute and to Vat 3 at a rate of 1ℓ per minute. The solution flows from Vat 2 to Vat 3 at a rate of 4ℓ per minute. Solution from Vat 3 is allowed to flow onto the ground at a rate of 6ℓ per minute (most likely destroying a fragile ecosystem, but that is none of our concern).

   (a) Write a system of equations that describes x_1, x_2 and x_3. (It may help to draw a diagram.)

   (b) Verify that $x_1(t) = 25e^{-t/20}$.

   (c) Find the solution for $x_2(t)$.

2. Consider the partially decoupled system

$$\frac{dx}{dt} = tx$$

$$\frac{dy}{dt} = txy.$$

   Solve the initial value problem where $x(0) = 1$ and $y(0) = e$.

3. Consider the linear system of differential equations

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x},$$

   where

$$A = \begin{bmatrix} 1 & 5 \\ 10 & -4 \end{bmatrix}.$$

   (a) Find the eigenvalues of A.

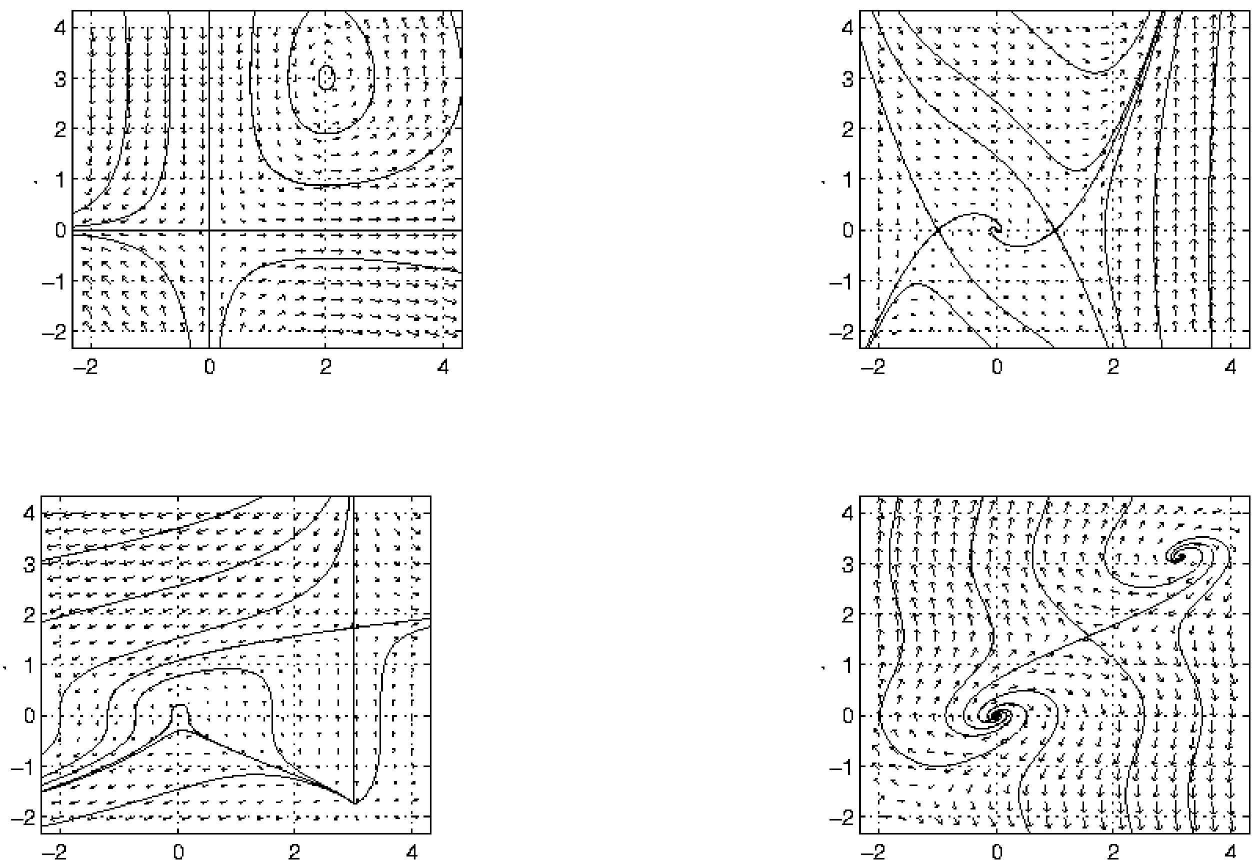
   (b) Find the eigenvectors of A.

   (c) Solve the initial value problem $\frac{d\mathbf{x}}{dt} = A\mathbf{x}$, $\mathbf{x}(0) = \begin{bmatrix} 0 \\ 9 \end{bmatrix}$.

4. Consider the system of differential equations,

$$\begin{aligned} \frac{dx}{dt} &= 3x - xy \\ \frac{dy}{dt} &= -4y + 2xy. \end{aligned}$$

(a) Circle the phase plane for this system:



(b) Find the equilibrium points of this system.

5. Consider the 3-dimensional system of linear differential equations

$$\frac{d\mathbf{x}}{dt} = \begin{bmatrix} x_1{}' \\ x_2{}' \\ x_3{}' \end{bmatrix} = \begin{bmatrix} -1 & 0 & 3 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix} \mathbf{x}.$$

(a) Find the eigenvalues for the system.

(b) Find the eigenvectors.

(c) Find the general solution to the system.

6. (a) Find the general solution to the linear, first-order system

$$\frac{dX}{dt} = \begin{bmatrix} 0 & -12 \\ 2 & -10 \end{bmatrix} X.$$

(b) Sketch some representative solution curves in the phase plane, and classify the equilibrium point as to type and stability.

7. Consider the linear system

$$\frac{dX}{dt} = \begin{bmatrix} 0 & -2 \\ -8 & 0 \end{bmatrix} X = AX.$$

(a) Find the general solution for the system using the eigenvalue method.

(b) Sketch some representative solutions in the phase plane.

8. Consider the system of differential equations,

$$\begin{aligned} \tfrac{dx}{dt} &= 3x - y \\ \tfrac{dy}{dt} &= -4y + 2x. \end{aligned}$$

(a) Find the equilibrium points of this system.

(b) Classify the equilibrium point at $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ as to type and stability.