21-260 DIFFERENTIAL EQUATIONS

PARTIAL FRACTIONS: A QUICK REFERENCE GUIDE

Partial fractions is a technique for expressing a rational function $f(x) = \frac{p(x)}{q(x)}$ as a sum of simpler rational functions. The main challenge is determining the form for the rational functions in the sum. It is also important, however, to make sure that the rational function f is *proper*, i.e. that the degree of the numerator is less than the degree of the denominator. If it is not, you must first write f as the sum of a polynomial and a proper rational function.

1. Proper Rational Functions

A proper rational function is one in which the degree of the numerator is less than the degree of the denominator. Any improper rational function can be written as the sum of a polynomial and a proper rational function. The key to this is the division algorithm for polynomials. For example, given the function

$$\frac{x^4 + 6x^2 - 2}{x^2 - x}$$

we perform the long division:

$$
\begin{array}{ccc|ccccc}
 & & & x^2 & +x & +7 & & \\
\cline{4-8}
x^2 & -1x & +0 & x^4 & +0x^3 & +6x^2 & +0x & -2 \\
 & & & \underline{x^4} & \underline{-x^3} & & & \\
 & & & & x^3 & +6x & +0x & -2 \\
 & & & & \underline{x^3} & \underline{-x^2} & & \\
 & & & & & 7x^2 & +0x & -2 \\
 & & & & & \overline{7x^2} & \overline{-7x} & \\
 & & & & & & 7x & -2
\end{array}
$$

This shows that we can write

$$\frac{x^4 + 6x^2 - 2}{x^2 - x} = x^2 + x + 7 + \frac{7x - 2}{x^2 - x}.$$

2. Choosing the Partial Fractions

The form that the partial fractions will take depends on the factors of the denominator $q(x)$. There are four main cases, which can be used in combination with each other.

2.1. **Case I: Unique Linear Factors.** If $q(x)$ has unique linear factors

$$q(x) = (x - a_1)(x - a_2) \ldots (x - a_n)$$

then the proper expression is

$$\frac{p(x)}{(x - a_1) \cdots (x - a_n)} = \frac{A_1}{x - a_1} + \cdots + \frac{A_n}{x - a_n}.$$

2.2. **Case II: Repeated Linear Factors.** If

$$q(x) = (x-a)^p$$

Then the proper expression is

$$\frac{p(x)}{(x-a)^m} = \frac{B_1}{x-a} + \frac{B_2}{(x-a)^2} + \cdots + \frac{B_m}{(x-a)^m}.$$

2.3. **Case III: Unique Quadratic Factors.** If $q(x)$ has unique irreducible quadratic factors

$$q(x) = (a_1x^2 + b_1x + c_1)\ldots(a_jx^2 + b_jx + c_j)$$

The the partial fractions take the form

$$\frac{p(x)}{(a_1x^2 + b_1x + c_1)\ldots(a_jx^2 + b_jx + c_j)} = \frac{C_1x + D_1}{a_1x^2 + b_1x + c_1} + \cdots + \frac{C_jx + D_j}{a_jx^2 + b_jx + c_j}$$

.

2.4. **Case IV: Repeated Quadratic Factors.** Finally, if $q(x)$ has a repeated irreducible quadratic factor

$$q(x) = (ax^2 + bx + c)^k$$

then the proper form is

$$\frac{p(x)}{(ax^2 + bx + c)^k} = \frac{C_1x + D_1}{(ax^2 + bx + c)} + \cdots + \frac{C_kx + D_k}{(ax^2 + bx + c)^k}$$

2.5. **Mixes Cases.** If the factors of $q(x)$ don't fall into any single category, then you must use several of the rules above, for example

$$\frac{p(x)}{(x-2)(x-3)^2(x^2+2)} = \frac{A}{x-2} + \frac{B_1}{x-3} + \frac{B_2}{(x-3)^2} + \frac{Cx+D}{x^2+2}.$$