21-260 **Differential Equations** **D. Handron**

Week #8 Written Assignment: Due on *Wednesday*, October 16.

1. When we discussed the method of undetermined coefficients for finding a solution to

$$ay'' + by' + cy = g(t),$$

we had three rules. "Rule 1" directed you to start with a guess of an linear combination (with undetermined coefficients) of all the terms that appear in $g(t)$ and it's derivatives. "Rule 2" said that if your first guess is a solution to the corresponding homogeneous equation, you should multiply your initial guess by t. This problem is intended to provide a partial justification for Rule 2.

Let $r, p \in \mathbb{R}$ and consider the equation

$$y'' - (r+p)y' + (rp)y = e^{rt}, \tag{1}$$

which has auxiliary equation $m^2 - (r+p)m + rp = (m-r)(m-p) = 0$. Assume that $r \neq p$. The Rule 1 guess for a particular solution would be $y = Ae^{rt}$, but this is s solution to the corresponding homogeneous equation. The steps of this problem guide you to finding the general solution.

   (a) We begin by looking for solutions of the form $y(t) = u(t)e^{rt}$: substitute $y = u(t)e^{rt}$ into equation (1) to find a differential equation that $u(t)$ must satisfy to ensure $y(t) = u(t)e^{rt}$ is a solution.

   (b) The equation you found in part (1a) is a linear second order equation, but it has no "zeroth-derivative" term, only u' and u'' terms, on the left hand side. You can integrate both sides to get a first order linear equation. Do that. What is the first order equation?

   (c) Find the general solution to the first order equation you found in part (1b).

   (d) What is the general solution to equation (1)? How does this expression compare to that you would get from using Rule 2 to find a particular solution and the triple formula to find the complementary solution?

2. The *gamma function* $\Gamma(\alpha)$ is defined by the improper integral

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t} dt, \quad \alpha > 0.$$

   (a) Use this definition to show that $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$.

   (b) Using the definition, compute ~~$\Gamma(1)$~~ $\Gamma(2)$.

   (c) Show that for positive integers n, $\Gamma(n+1) = n!$. [Note: you can use *mathematical induction* — it is enough to show that $\Gamma(2) = 1!$ and that if we assume $\Gamma(k+1) = k!$ it follows that $\Gamma(k+2) = (k+1)!$.]

3. Use the previous problem and a change of variables to obtain the result

$$\mathscr{L}\{t^\alpha\} = \frac{\Gamma(\alpha+1)}{s^{\alpha+1}}, \quad \alpha > -1.$$

Note that for positive integers n this implies that

$$\mathscr{L}\{t^n\} = \frac{n!}{s^{n+1}}.$$

4. (a) Recall that $\mathscr{L}\{f'(t)\} = s\mathscr{L}\{f(t)\} - f(0)$. Using this result with $f(t) = te^{at}$, evaluate $\mathscr{L}\{te^{at}\}$.

   (b) Recall that $\mathscr{L}\{f''(t)\} = s^2\mathscr{L}\{f(t)\} - sf(0) - f'(0)$. Using this result with $f(t) = t\sin(kt)$, evaluate $\mathscr{L}\{t\sin(kt)\}$. It may help to remember that $\mathscr{L}\{\sin kt\} = \frac{k}{s^2+k^2}$ and $\mathscr{L}\{\cos kt\} = \frac{s}{s^2+k^2}$.

   (c) Evaluate $\mathscr{L}\{t\cos(kt)\}$.