**Week #7 Written Assignment**: Due on Friday, October 11.

1. **(Method of undetermined coefficients III)** Solve the initial value problem

$$y'' + 10y' + 25y = e^{-5x}, \quad y(0) = 0,\, y'(0) = 0$$

2. Assume that the system described by the equation

$$mx'' + \beta x' + kx = 0$$

is critically damped and that the initial conditions are $x(0) = x_0$ and $x'(0) = v_0$.

   (a) Show that if $x_0 \neq 0$ and $v_0 = 0$ then $x \to 0$ as $t \to \infty$ but that $x(t) \neq 0$ whenever $t > 0$.

   (b) If $x_0$ is positive, determine a condition on $v_0$ that will ensure that the mass passes through its equilibrium position after it is released.

3. **Logarithmic Decrement.** A solution to the equation

$$mx'' + \beta x' + kx = 0$$

can be written in the form

$$x(t) = Ae^{-\lambda t}\sin(\mu t + \phi)$$

where $\lambda = \frac{\beta}{2m}$ and $\mu = \sqrt{\omega^2 - \lambda^2}$. (Recall that $\omega^2 = \frac{k}{m}$.)

   (a) For the damped oscillation described by $x(t) = Ae^{-\lambda t}\sin(\mu t + \phi)$ show that the time between successive maxima is $T_d = 2\pi/\mu$. [Note: because of the exponential term, the maxima of $x(t)$ do not occur at the same time as the maxima of $\cos(\mu t - \delta)$.]

   (b) Show that the ratio of the displacements at two successive maxima is given by

$$e^{\lambda T_d}.$$

   Observe that this ratio does not depend on which pair of maxima is chosen. The natural logarithm of this ratio is called the logarithmic decrement and is denoted by $\Delta$.

   (c) Show that $\Delta = \frac{\pi\beta}{m\mu}$.

Since $m$, $\mu$, and $\Delta$ are quantities that cam be measured easily for a mechanical system, this result provides a convenient and *practical* method for determining the damping constant of the system, which is more difficult to measure directly.

4. When we discussed second order, linear, homogeneous, constant-coefficient equations we first found exponential solutions by finding roots of the auxiliary equation, $am^2+bm+c=0$. When the equation had a double root, i.e. $am^2+bm+c=a(m-r)^2$, we needed to find a second solution to form a fundamental set of solutions. We verified that in that case, $y(t)=te^{rt}$ is a solution, but without any motivation for why that is a reasonable thing to consider. The purpose of this problem is to provide motivation for that choice.

   We'll consider the example

$$y''+4y'+4y=0. \tag{1}$$

   (a) Using the auxiliary equation, show that the only exponential solutions to (1) are $c_1e^{-2t}$.

   (b) Justify the claim: any solution $y(t)$ to equation (1) can be written as $y(t)=u(t)e^{-2t}$ for some function $u(t)$.

   (c) Substitute $y=u(t)e^{-2t}$ into equation (1) to find a differential equation that $u(t)$ must satisfy if $u(t)e^{-2t}$ is a solution.

   (d) Solve that differential equation to find all possible functions $u(t)$

   (e) Use the results above to write the general solution to equation (1)