21-260 **Differential Equations** **D. Handron**

Week #3 Written Assignment: Due on Friday, February 1.

1. Consider a vat that at time t contains a volume $V(t)$ of salt solution containing an amount $Q(t)$ of salt, evenly distributed throughout the vat with a concentration $c(t)$, where $c(t) = Q(t)/V(t)$. Assume that water containing a concentration k of salt enters the vat at a rate r_{in}, and that water is drained from the vat at a rate $r_{out} > r_{in}$.

   (a) If $V(0) = V_0$, find an expression for the amount of solution in the vat at time t. At what time T will the vat become empty? Find an initial value problem that describes the amount of salt in the vat at time $t \leq T$. You may assume that $Q(0) = Q_0$.

   (b) Solve the initial value problem in part (a). What is the amount of salt in the vat at time t?What is the concentration of the last drop that leaves the vat at time $t = T$?

2. One model of technological innovation in the agriculture industry assumes that a farmer will adopt an innovation (e.g. an improved method of harvesting) only after learning of it from another farmer who has already adopted the innovation.

   Consider a community of farmers, and let p denote the fraction of these farmers who have adopted a particular innovation. The above assumption leads to the model

   $$\frac{dp}{dt} = kp(1-p)$$

   (where $0 \leq p \leq 1$) for the rate of adoption of the innovation.

   (a) Explain why it is reasonable to assume that the rate of adoption is proportional to both p and to $(1-p)$.

   (b) Find the constant solutions for this differential equation. Explain these solutions in terms of the model.

   (c) Assume that at time $t = 0$ a fraction F of the farmers have adopted the innovation. Find and explicit function $p(t)$ that gives the fraction of farmers who have adopted the innovation at time t.

   (d) Sketch the direction field for this equation ($0 \leq p \leq 1$). Use the direction field to sketch some solution curves.

   (e) Does the long term behavior (as $t \to \pm\infty$) of the solutions you computed in part (2c) agree with the curves sketched in part (2d)?

3. (Note: Don't make this problem harder than it really is, which is not too hard. It is a foreshadowing of things to come.)

(a) Show that the solution

$$y(t) = \frac{1}{\mu(t)} \left[C + \int_{t_0}^{t} \mu(s) g(s)\, ds \right]$$

of the differential equation

$$y' + p(t)y = g(t)$$

can be written in the form $y(t) = Cy_1(t) + y_2(t)$. Identify the functions $y_1(t)$ and $y_2(t)$.

(b) Show that $y_1(t)$ is a solution to the homogeneous differential equation

$$y' + p(t)y = 0.$$

(c) Show that $y_2(t)$ is a solution to the original, non-homogeneous, differential equation

$$y' + p(t)y = g(t).$$

4. Consider the initial value problem

$$\frac{dy}{dt} = \frac{-t + (t^2 + 4y)^{1/2}}{2}, \quad y(2) = -1.$$

(a) Verify that both $y_1(t) = 1 - t$ and $y_2(t) = -t^2/4$ are both solutions to the initial value problem.

(b) Explain why the existence of two solutions doesn't violate the uniqueness part of the existence and uniqueness theorems discussed in class.

(c) Show that for any $c \in \mathbb{R}$, the function $y(t) = ct + c^2$ is a solution to the differential equation for $t \geq -2c$; that if $c = -1$, then the solution $y(t)$ satisfies the initial condition, and that in this case $y(t) = y_1(t)$; and that there is no choice of c that makes $y(t)$ equal to $y_2(t)$.