Week #2 Written Assignment: Due on Wednesday, May 27.

1. Consider a vat that at time t contains a volume $V(t)$ of salt solution containing an amount $Q(t)$ of salt, evenly distributed throughout the vat with a concentration $c(t)$, where $c(t) = Q(t)/V(t)$. Assume that water containing a concentration k of salt enters the vat at a rate r_{in}, and that water is drained from the vat at a rate $r_{out} > r_{in}$.

   (a) If $V(0) = V_0$, find an expression for the amount of solution in the vat at time t. At what time T will the vat become empty? Find an initial value problem that describes the amount of salt in the vat at time $t \leq T$. You may assume that $Q(0) = Q_0$.

   (b) Solve the initial value problem in part (a). What is the amount of salt in the vat at time t?What is the concentration of the last drop that leaves the vat at time $t = T$?

2. Consider the initial value problem

$$\frac{dy}{dt} = \frac{-t + (t^2 + 4y)^{1/2}}{2}, \quad y(2) = -1.$$

   (a) Verify that both $y_1(t) = 1 - t$ and $y_2(t) = -t^2/4$ are both solutions to the initial value problem.

   (b) Explain why the existence of two solutions doesn't violate the uniqueness part of the existence and uniqueness theorems discussed in class.

   (c) Show that for any $c \in \mathbb{R}$, the function $y(t) = ct + c^2$ is a solution to the differential equation for $t \geq -2c$; that if $c = -1$, then the solution $y(t)$ satisfies the initial condition, and that in this case $y(t) = y_1(t)$; and that there is no choice of c that makes $y(t)$ equal to $y_2(t)$.

3. Consider the differential equation

$$y' = t^2 - y^2$$

   (a) Sketch the direction field for this equation. Show the isoclines for $m = 0, 1, -1$ and whatever other information is needed to give a good idea of how solutions behave.

   (b) I claim that the solution satisfying the initial condition $y(1) = 1/2$ also satisfies $y(t) > 0$ for *every* $t > 1$. Is this assertion correct? Justify your answer.

   (c) I claim that the solution satisfying $y(1) = 1/2$ also satisfies $y(t) > t$ for *some* $t > 1$. Is this assertion correct? Justify your answer.

4. A differential equation of the form

$$\frac{dy}{dx} + P(x)y = Q(x)y^n, \quad n \in \mathbb{R}$$

is called a *Bernoulli Equation*. Note that for $n \in \{0, 1\}$ the Bernoulli equation is linear.

(a) Assume that $n \neq 0$ and $n \neq 1$. Show that the substitution $u = y^{1-n}$ results in a linear differential equation for u. [Hint: divide both sides of the Bernoulli equation by y^n.]

(b) Solve the Bernoulli equation $\frac{dy}{dx} + 8y = e^{2x}y^5$.