Week #13 Written Assignment: Due on *Friday*, November 22.

1. Consider the 2-dimensional system of linear equations

$$X' = \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix} X$$

   (a) Determine the eigenvalues of the system. Note that it is a repeated eigenvalue. Find all corresponding eignevectors.

   (b) Classify the equilibrium point of the system (by type and stability).

   (c) Find the nullclines for the system and determine where in the phase plane solutions are traveling Up/Down and Left/Right.

   (d) Using information from the eigenvalues and eigenvectors and nullclines, draw a phase portrait for the system.

2. Consider the 2-dimensional system of linear equations

$$X' = \begin{bmatrix} 1 & 4 \\ -1 & 1 \end{bmatrix} X$$

   (a) Determine the eigenvalues of the system. Note that eigenvalues are complex.

   (b) Classify the equilibrium point of the system (by type and stability).

   (c) Find the nullclines for the system and determine where in the phase plane solutions are traveling Up/Down and Left/Right.

   (d) Using information from the eigenvalues and nullclines, draw a phase portrait for the system.

3. Consider the 2-dimensional system of linear equations

$$X' = \begin{bmatrix} \alpha & -2 \\ 2 & \alpha \end{bmatrix} X$$

   Note that the coefficient matrix for this system contains a parameter α.

   (a) Determine the eigenvalues of the system in terms of α.

   (b) The qualitative behavior of the solutions depends on the value of α. Determine a value α_0 where the qualitative behavior changes (i.e. the classification of the equilibrium point at the origin differs depending on whether $\alpha > \alpha_0$ or $\alpha > \alpha_0$). Classify the equilibrium point of the system (by type and stability) when $\alpha < \alpha_0$, when $\alpha = \alpha_0$, and when $\alpha > \alpha_0$.

(c) Sketch a sequence of 5 phase portraits for the system, for the follwoing 5 situations: $\alpha \ll \alpha_0$, $\alpha \lesssim \alpha_0$, $\alpha = \alpha_0$, $\alpha \gtrsim \alpha_0$, $\alpha \gg \alpha_0$.

Note: the symbols $\ll$ and $\gg$ mean "much less than" and "much greater than" while $\lesssim$ and $\gtrsim$ mean " less than but close to" and "greater than but close to."

4. Consider the 2-dimensional system of linear equations

$$X' = \begin{bmatrix} -2 & 1 \\ -\alpha & -2 \end{bmatrix} X$$

Note that the coefficient matrix for this system contains a parameter α.

(a) Determine the eigenvalues of the system in terms of α.

(b) The qualitative behavior of the solutions depends on the value of α. It changes from one type to another at $\alpha = 0$ (also when $\alpha = -4$, but this problem focuses on the change at 0). Classify the equilibrium point of the system (by type and stability) when $-4 < \alpha < 0$, when $\alpha = 0$, and when $\alpha > 0$.

(c) Determine the eigenvectors for the system (in terms of α) for the cases $-4 < \alpha < 0$ and $\alpha = 0$.

(d) Determine the nullclines for the system (in terms of α) in the case where $\alpha > 0$.

(e) Using information from the eigenvalues and eigenvectors, draw a sequence of 5 phase portraits for the system, for the follwoing 5 situations: $-4 < \alpha < 0$, $\alpha \lesssim 0$, $\alpha = 0$, $\alpha \gtrsim 0$, $\alpha \gg 0$,

5. The purpose of this problem is to show a connection between second order linear equations and first order systems of equations. It also provides an example of why linear systems of differential equations are worth studying (assuming you think second order linear equations are worthwhile).

Consider the second order linear equation

$$ax'' + bx' + cx = 0. \tag{1}$$

(a) Introduce a new variable, v, by setting $v = x'$. Find an expression for v' in terms of x and v.

(b) Show that we can express equation (1) as an equivalent 2-dimensional system of first-order of equations

$$\frac{d}{dt}\begin{bmatrix} x \\ v \end{bmatrix} = A \begin{bmatrix} x \\ v \end{bmatrix}. \tag{2}$$

What is the coefficient matrix A for this system?

(c) In finding the solutions to a first-order linear system, the first step is to find the eigenvalues by setting $\det(A - \lambda I) = 0$. What is interesting about the equation $\det(A - \lambda I) = 0$? Where have you seen this before?

(d) In the case where $a = 1$, $b = 5$, and $c = 6$, find the solution to the system (2).

(e) In the same case ($a = 1$, $b = 5$, and $c = 6$) find the solution to equation (1) by using the triple formula.

(f) How is the solution you found in part (5e) related to the solution you found in (5d)?