Week #10 Written Assignment: Due on *Friday*, November 1.

1. (a) Show that for $c > 0$ and $t \neq c$

$$\mathscr{U}(t-c) = \int_0^t \delta(u-c)\,du.$$

   (b) Show that for $c > 0$

$$\mathscr{L}\{\delta(t-c)\} = s\,\mathscr{L}\{\mathscr{U}(t-c)\} - \mathscr{U}(-c).$$

   (c) A friend of yours looks at the results of (a) and (b) and exclaims "Oh, the delta function is basically the derivative of the unit step function." Why would your friend say this? Does the statement make sense in terms of rates of change or slopes of graphs?

2. (a) Find the solution to the differential equation

$$y'' + y = \sum_{k=1}^{20}(-1)^{k+1}\delta(t - \frac{k\pi}{2})$$

   subject to the initial conditions $y(0) = 0$ and $y'(0) = 0$.

   (b) sketch a plot of the solution for $0 < t < 3\pi$.

   (c) Give an interpretation of the results in terms of the behavior of a mass-spring system being struck with a hammer.

3. Consider the system of differential equations

$$\begin{aligned} \tfrac{dx}{dt} &= 1 - x - y \\ \tfrac{dy}{dt} &= 1 - x^2 - y^2 \end{aligned}$$

   (a) Sketch the x-nullcline, where solutions must travel vertically. Identify the regions in the plane where solutions will move toward the right, and where solutions move toward the right.

   (b) On a separate set of axes, sketch the y-nullcline, where solutions must travel horizontally. Identify the regions in the plane where solutions will move upward, and where solutions move downward.

   (c) On a third set of axes, sketch both the x- and y-nullclines. Indicate the regions in the plane where solutions travel up-right, up-left, down-left, and down-right.

   (d) On a fourth set of axes, sketch both the x- and y-nullclines. Sketch trajectories (solution curves) passing through the following points:

$$\left(\frac{1}{2}, \frac{1}{2}\right), \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \left(\frac{1}{2}, -\frac{1}{2}\right), \left(\frac{3}{2}, -\frac{1}{2}\right)$$

   (e) Suppose that $(x(t), y(t)$ is a solution satisfying the initial condition $x(0) = x_0$, $y(0) = y_0$, where the point (x_0, y_0) lies above the line $y = 1-x$ and inside the circle $x^2+y^2 = 1$. What are $\lim_{t\to\infty} x(t)$ and $\lim_{t\to\infty} y(t)$?