Closed book and closed notes; Calculators not permitted.

1. **(25 points)** Find the general solution to the following differential equations:

   (a) $\frac{dy}{dt} = t + ty + t^2 + t^2y$

   (b) $x\frac{dy}{dx} = 2y + x^3\cos(x)$

2. **(25 points)** The logistic equation $\frac{dP}{dt} = P(10 - P)$ can be used to model the growth of a population with limited resources, a population of fruit flies, for instance. (In this case, we will measure the population P in thousands of fruit flies, and t in days.)

   (a) Draw the phase line (or phase diagram) for this equation. What are the equilibrium points? Describe the behavior of the non-equilibrium solutions.

   (b) Modify this equation to account for "harvesting" of the fruit flies, at a rate of 9000 flies per day. Draw the phase line (or phase diagram) for the modified system. What are the equilibrium solutions? Describe the behavior of the non-equilibrium solutions.

3. **(25 points)** Consider the first order linear equation

$$e^t\frac{dy}{dt} + \sin(t)y = 0.$$

   (a) Is there a solution to this equation defined for $0 \leq t < \infty$ with initial conditions $y(0) = 1$? How do you know?

   (b) How many such solutions are there?

4. **(25 points)** Let's consider a model of the federal budgeting process. Assume that Congress is divided into three groups. The Budget Cutters, the Tax and Spenders and the Status Quo-ers. When there is a budget surplus, the Tax and Spenders attempt to increase the number of social programs with an effort proportional to the size of the surplus. When there is a deficit (or negative surplus) the Budget Cutters attempt to reduce spending, also in proportion to the size of the dificit (and with the same constant of proportionality.) The Status Quo-ers oppose any change in the budget, and are most active when the changes in the budget are greatest (i.e. when the rate of change of the surplus is greatest.) The size of the suprlus can thus be modeled by a second order differential equation:

$$\frac{d^2S}{dt^2} = -5S - 4\frac{dS}{dt},$$

   where -5 is the proportionality constant for the Budget Cutters and Tax and Spenders, and -4 is the proportionality constant of the Status Quo-ers. What does this model predict about the future of budget surplusses and deficits? (Negative values of S correspond to budget deficits.)

5. **(25 points)** Consider the 3-dimensional system of linear differential equations

$$\frac{d\mathbf{x}}{dt} = \begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} = \begin{bmatrix} -1 & 0 & 3 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix} \mathbf{x}.$$

(a) Find the eigenvalues for the system.

(b) Find the eigenvectors.

(c) Find the general solution to the system.

6. **(25 points)** Use Laplace transforms to solve the following non-homogenious equations:

(a) $x'' + 9x = \cos(4t)$, where $x(0) = 0$ and $x'(0) = 0$. It may help to know that $\cos(A)\sin(B) = \frac{1}{2}[\sin(A+B) - \sin(A-B)]$.

(b) $x'' + 9x = 3\delta(t-3)$, with $x(0) = 1$ and $x'(0) = 0$.

7. **(25 points)** Compute the Fourier Series of the periodic function f with period 4 such that

$$f(t) = t + 1 \quad \text{for} \quad -2 \leq t \leq 2.$$

8. **(25 points)** The goal of this problem is to solve the wave equation

$$\frac{\partial^2 y}{\partial t^2} = 9\frac{\partial^2 y}{\partial x^2}$$

subject to the conditions $y(0,t) = y(2,t) = 0$, $y(x,0) = \sin(\frac{\pi t}{2})$ and $\frac{\partial y}{\partial t}(x,0) = 0$. Begin by looking for a solution of the form $y(x,t) = X(x)T(t)$.

(a) Solve $X'' + \lambda X = 0$, such that $X(0) = X(2) = 0$. For what values of λ does such a solution exist? Why did we require the endpoint conditions $X(0) = X(2) = 0$?

(b) For each λ in part (a), find a solution to $T'' + 4\lambda T = 0$, such that $T'(0) = 0$. Why do we require that $T'(0) = 0$?

(c) For each λ_n in (a) you get a solution $y_n(x,t) = X_n(t)T_n(t)$. write $y(x,t) = \sum_{n=1}^{\infty} A_n y_n(x,t)$. Find the coefficients A_n needed to satisfy the condition $y(x,0) = \sin(\frac{\pi t}{2})$.