Homework #5: Due on Wednesday, September 26

1. Consider the system of equations

$$\begin{array}{rcrcrcrcr} & & & & 2x_3 & + & 6x_4 & = & -4 \\ x_1 & - & 2x_2 & & & + & x_4 & = & 4 \\ 2x_1 & - & 4x_2 & + & 2x_3 & + & 8x_4 & = & 4. \end{array}$$

   (a) What is the augmented matrix for this system?

   (b) Use elementary row operations to put the matrix into reduced (row) echelon form. Perform one operation at at time and indicate clearly which row operations are used.

   (c) Write the solution to the system in vector form.

2. Is the vector $\begin{bmatrix} -5 \\ 3 \\ -1 \end{bmatrix}$ a linear combination of the vectors

$$\begin{bmatrix} 4 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 11 \\ -1 \\ -2 \end{bmatrix}?$$

   (a) Write down a system of linear equations in standard form whose solution will provide an answer to this question.

   (b) Solve the system from part (a). Either give a description of all such linear combinations, or explain how you know there are none.

3. (a) A *dilation* is a function $D : \mathbb{R}^n \to \mathbb{R}^n$ given by $D(\mathbf{x}) = r\mathbf{x}$ for some $r \in \mathbb{R}$. Show that every dilation is a linear transformation. It should be clear what theorems, properties, or axioms are applied in your argument.

   (b) A (counterclockwise) *rotation* through an angle θ in $\mathbb{R}^2$ is a linear transformation $R : \mathbb{R}^2 \to \mathbb{R}^2$ satisfying

$$R\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}, \quad R\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} \cos\theta - \sin\theta \\ \cos\theta + \sin\theta \end{bmatrix},$$

   Find a matrix A_θ such that $R(\mathbf{x}) = A_\theta\mathbf{x}$ for every $\mathbf{x} \in \mathbb{R}^2$. It should be clear what theorems, properties, or axioms are applied in your argument.

4. (a) Write vector space axiom #4 in "symbolic" notation, i.e. using the symbols $\forall, \exists, \in$, and so on. Use V to denote the vector space in question. Choose appropriate symbols for vectors in V.

   (b) Find the negation of the expression you found in part (a). Again, use the symbols $\forall, \exists, \in$, as needed, V for the vector space "candidate," and appropriate symbols for vectors in V. Work through the negation step-by-step, or make clear in some other way how you arrived at your result.

(c) Let $\mathbf{a}$ and $\mathbf{b}$ be non-zero vectors in $\mathbb{R}^n$, neither one a multiple of the other. Show that the set $\{\mathbf{a} + t\mathbf{b} : t \in \mathbb{R}\}$ is not a vector space by demonstrating that the statement you found in part (b) (the negation of axiom 4) is *true*.