21-241 **Matrix Algebra** **D. Handron**

Exam #1 Review
February 16, 2005 **Name:**______________________________

Closed book and notes; calculators not permitted.

1. Let
$$A = \begin{bmatrix} 1 & -2 & 0 \\ 1 & -3 & 1 \\ 0 & -1 & 1 \end{bmatrix}$$
   (a) For which vectors $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ does the equation $A\mathbf{x} = \mathbf{b}$ have a solution?

   (b) Find all solutions of the homogeneous equation $A\mathbf{x} = \mathbf{0}$.

2. Let
$$A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \\ 3 & 3 & 1 \end{bmatrix}$$
   Solve the linear system $A\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

3. Let $R(\mathbf{x})$ be a $60°$ rotation in the counter-clockwise direction. Find the matrix B such that $R(\mathbf{x}) = B(\mathbf{x})$.

4. (a) Find all solutions to the equation $A\mathbf{x} = \mathbf{b}$, where
$$A = \begin{bmatrix} 1 & 1 & 1 & 5 & 0 \\ 0 & 1 & -1 & 2 & 0 \\ 2 & 0 & 4 & 6 & 1 \\ 1 & 2 & 0 & 7 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ 5 \\ 1 \end{bmatrix}$$
   (b) Let $T : \mathbb{R}^5 \to \mathbb{R}^4$ be the linear transformation defined by $T(\mathbf{x}) = A\mathbf{x}$. Find all solutions of the equation $T(\mathbf{x}) = \mathbf{0}$. (Write your answer as a span of vectors if you can.)

5. $T : \mathbb{R}^2 \to \mathbb{R}^2$ is the linear transformation such that

$$T\left(\begin{bmatrix} 2 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} -4 \\ 7 \end{bmatrix}$$

and

$$T\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$$

(a) What is the matrix for this linear transformation? (Hint: $\begin{bmatrix} 1 \\ 0 \end{bmatrix} = 2\begin{bmatrix} 2 \\ 3 \end{bmatrix} - 3\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 2 \\ 3 \end{bmatrix}$.)

(b) Interpret the action of T geometrically.

6. Let T and L be linear transformations such that

$$T(\mathbf{x}) = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}\mathbf{x} \quad \text{and} \quad L\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad L\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}.$$

What is the matrix for the composite transformation $L(T(\mathbf{x}))$?

7. (a) Describe the solutions of the vector equation

$$x_1\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} + x_2\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + x_3\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} + x_4\begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

(b) Describe the solution set of the system of equations

$$\begin{array}{lllll} 3x_1 & +x_2 & & +2x_4 & = 0 \\ 2x_1 & +x_2 & -x_3 & & = 0 \\ x_1 & & +x_3 & +2x_4 & = 0 \end{array}$$

Write the solution in parametric vector form if possible.

8. Let

$$A = \begin{bmatrix} 1 & 1 & -2 & -2 & 1 \\ 0 & 1 & 2 & 0 & -2 \\ 2 & 3 & -2 & -4 & 3 \\ -1 & -1 & 2 & 2 & -3 \end{bmatrix}.$$

(a) Use row operations to find the reduced echelon form of A.

(b) Is A row equivalent to the matrix

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}?$$

Why or why not?

9. Consider the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -2 \\ -1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} -1 \\ 2 \\ 0 \\ -1 \end{bmatrix}, \mathbf{v}_4 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}.$$

(a) Do the vectors $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ form a linearly independent set? Why or why not?

(b) Do the vectors $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ form a linearly independent set? Why or why not?

10. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation such that

$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \quad \text{and} \quad T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

and let $R : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation defined by

$$R(\mathbf{x}) = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{3}{\sqrt{2}} \end{bmatrix} \mathbf{x}.$$

(a) Find a matrix A such that $T(\mathbf{x}) = A\mathbf{x}$.

(b) Find a matrix B such that

$$(R \circ T)(\mathbf{x}) = R(T(\mathbf{x})) = B\mathbf{x}.$$

(c) Draw a diagram that shows the image of the unit square under the transformation $R \circ T$. How would you describe this transformation in terms of "simple" linear transformations (reflections, rotations, contractions/dilations, shears)?

11. Suppose that A is a 3×3 matrix and $\mathbf{y}$ is a vector in $\mathbf{R}^3$ such that the equation $A\mathbf{x} = \mathbf{y}$ does *not* have a solution.

Does there exist a vector $\mathbf{z}$ in $\mathbf{R}^3$ such that the equation $A\mathbf{x} = \mathbf{z}$ has a unique solution? Why or why not? [Hint: Think about the possible reduced echelon forms for A.]

12. Suppose

$$(AB)^{-1} = \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix} \quad \text{and} \quad B^{-1} = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}.$$

(a) Find A.

(b) Is it true that $AB = BA$?

13. Let

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \\ 3 & 3 & 1 \end{bmatrix}$$

(a) Find A^{-1}.

(b) Solve the linear system $A\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ by using the result of part (a). (Hint $\mathbf{x} = A^{-1}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.)

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(a) What is the matrix for this linear transformation? (Hint: $\begin{bmatrix} 1 \\ 0 \end{bmatrix} = 2\begin{bmatrix} 2 \\ 3 \end{bmatrix} - 3\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 2 \\ 3 \end{bmatrix}$.)

(b) Is the linear transformation T invertible? If so, what is the inverse transformation?

15. A, B and C are 2×2 matrices such that

$$ABC = \begin{bmatrix} 5 & -8 \\ 7 & -11 \end{bmatrix} \quad \text{and} \quad (BC)^{-1} = \begin{bmatrix} -3 & 2 \\ -2 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$$

What are the matrices A and C?