A Calculus Example

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Abstract

This is a worked example from page 260 of Stewart's calculus text *Single Variable Calculus, Early Transcendentals*, $4^t h$ edition, published by Brooks-Cole.

Linear Approximations and Differentials

Example

Find the linearization of the function $f(x) = \sqrt{x+3}$ at $a = 1$ and use it to approximate the numbers $\sqrt{3.98}$ and $\sqrt{4.05}$. Are these approximations overestimates or underestimates?

Solution

The derivative of $f(x) = (x+3)^{1/2}$ is

$$f'(x) = \frac{1}{2}(x+3)^{-1/2} = \frac{1}{2\sqrt{x+3}}$$

so we have $f(1) = 2$ and $f'(1) = 1/4$. Putting these values into the equation for the linear Taylor polynomial gives

$$\sqrt{x+3} \simeq L(x) = f(1) + f'(1)(x-1) = 2 + \frac{1}{4}(x-1) = \frac{7}{4} + \frac{x}{4}.$$

In particular we have

$$\sqrt{3.98} \simeq \frac{7}{4} + \frac{0.98}{4} = 1.995 \text{ and } \sqrt{4.05} \simeq \frac{7}{4} + \frac{1.05}{4} = 2.0125.$$

	x	$L(x)$	Actual Value
$\sqrt{3.9}$	0.9	1.075	1.974841...
$\sqrt{4}$	1	2	2.000000...
$\sqrt{4.1}$	1.1	2.025	2.024845...
$\sqrt{6}$	3	2.5	2.449489...

Figure 1: Comparison of $f(x) = \sqrt{x+3}$ and its linear Taylor polynomial centered at $x = 1$

Since the square root function is concave down near $x = 1$, the tangent line lies above the graph of f, so these are both overestimates.

Figure 1 compares estimates from the linear approximation with the true values. Notice that the approximation gives good estimates for x near 1 and the accuracy deteriorates when x is farther away from 1.

Hints

The symbol $\simeq$ is made with `\simeq` (in math mode).