21-260: Differential Equations
QUIZ 10

Solutions

1. Write the Fourier series for the function

$$f(x) = Ax + B, \qquad -1 \le x \le 1; \qquad\qquad f(x+2) = f(x).$$

Here A and B are constants.

Here $L = 1$.

Notice that $f(x)$ is the sum of the odd function $g(x) = Ax$ and even function $h(x) = B$. The Fourier series of $g(x) = Ax$ is a sine series, The Fourier series of $h(x) = B$ is a cosine series. The cosine series of $h(x) = B$ is B. The sine series of $g(x) = Ax$ when $L = 1$ is

$$\sum_{n=1}^{\infty} b_n \sin(n\pi x),$$

where

$$b_n = 2\int_0^1 g(x)\sin(n\pi x)dx = 2A\int_0^1 x\sin(n\pi x)dx =$$
$$= -\frac{2A}{n\pi}(\cos(n\pi) - \int_0^1 \cos(n\pi x)dx) = \frac{2A(-1)^{n+1}}{n\pi}.$$

Answer: $f(x) = B + \sum_{n=1}^{\infty} \frac{2A(-1)^{n+1}}{n\pi}\sin(n\pi x)$.

Please see the second problem on the back!

2. Solve the heat conduction problem:

$$\begin{aligned} u_{xx} &= 9u_t, \qquad 0 < x < \pi, \quad t > 0; \\ u(0,t) &= 10, \qquad u(\pi,t) = 10, \quad t > 0; \\ u(x,0) &= \sin(x) + 2\sin(2x), \qquad 0 < x < \pi. \end{aligned}$$

Here $\alpha = 1/3$, $L = \pi$ and the steady-state temperature is $T = 10$.

The solution is

$$u(x,t) = 10 + \sum_{n=1}^{\infty} c_n e^{-n^2 t/9} \sin(nx),$$

where c_n is the coefficient of $\sin(nx)$ in $u(x,0) = \sin(x) + 2\sin(2x) - 10$.

Let's write the sine series of 10 over the interval $[0,\pi]$.

$$10 = \sum_{n=1}^{\infty} b_n \sin(nx),$$

where

$$\begin{aligned} b_n &= \frac{2}{\pi}\int_0^{\pi} 10\sin(nx)dx = \frac{20}{\pi n}(1-\cos(n\pi)) = \\ &= \frac{20(1-(-1)^n)}{\pi n} = \begin{cases} 0, & n \text{ is even,} \\ \frac{40}{\pi n}, & n \text{ is odd.} \end{cases} . \end{aligned}$$

Thus

$$\sin(x) + 2\sin(2x) - 10 = \sin(x) + 2\sin(2x) - \sum_{n=1}^{\infty} \frac{40}{\pi(2n-1)} \sin(2n-1)x =$$

$$= (1 - \frac{40}{\pi})\sin(x) + 2\sin(2x) - \sum_{n=1}^{\infty} \frac{40}{\pi(2n+1)} \sin(2n+1)x.$$

Answer:

$$u(x,t) = 10 + (1 - \frac{40}{\pi})e^{-t/9}\sin(x) + e^{-4t/9}\sin(2x) - \sum_{n=1}^{\infty} \frac{40}{\pi(2n+1)} e^{-(2n+1)^2 t/9} \sin(2n+1)x.$$