Last Name: ______________________________

First Name: ______________________________

Differential Equations Final Exam Practice

Please find out in what section you are and practice circling it!!!

Chris Almost Section A	Chris Almost Section B	Elena Canepa Section C	Tankut Dogrul Section D	Chris Almost Section E

Raja Ahmad Section F	Elena Canepa Section G	Raja Ahmad Section H	Tankut Dogrul Section I

Disclaimer: the only way in which this exam is similar to the actual Final Exam is the number of problems and their difficulty. The topics and the type of the problems in the actual Final Exam might be different.

1. A tank originally contains 10 gal of water with 1/2 lb of salt in solution. Water containing a salt concentration of $\frac{1}{200}(10-t)^2(\sin(t)+1)$ lb per gallon flows into the tank at a rate of 1 gal/ min, and the mixture is allowed to flow out of the tank at a rate of 2 gal/ min. The mixture is kept uniform by stirring. Let $Q(t)$ (in lb) be the amount of salt in the tank after time t (in min).

(i) How long (in min) will it take for the tank to become empty?

Answer:

(ii) Write the initial value problem for $Q(t)$ (before the tank is empty) and solve it.

Answer: $Q(t) =$

2. Solve the differential equation

$$(2xy + y^3)dx + (x^2 + 3xy^2 - 2y)dy = 0.$$

Answer:

3. Find the general solution of

$$\frac{d\bar{x}}{dt} = \begin{pmatrix} 4 & 2 \\ -1 & 6 \end{pmatrix} \bar{x}.$$

Answer: $\bar{x} =$

(ii) Draw the phase portrait of the system. What is the origin called in this case? Is the origin stable, unstable or semi-stable?

Answer:

4. Use **the method of undetermined coefficients** to solve the initial value problem

$$y'' - 4y' + 13y = (4t-4)\cos(3t) + (12t-6)\sin(3t), \qquad y(0) = 1, \quad y'(0) = 1.$$

Answer:

5. Find the solution of the given initial value problem.

$$y'' + 4y = g(t), \qquad y(0) = 0, \qquad y'(0) = 0,$$

where

$$g(t) = \begin{cases} 1, & t < \pi, \\ \sin(t), & t \geq \pi. \end{cases}$$

Answer:

6. (i) Show that the Laplace transform of $f(t) = t\sin(t)$ is

$$\frac{2s}{(s^2+1)^2}.$$

Hint: Use the relationship between the Laplace transform of a function and that of its derivative.

(ii) Use **the Laplace transform** and (i) to solve the initial value problem

$$y'' + y = \cos(t), \qquad y(0) = 0, \quad y'(0) = 1.$$

Answer:

7. Let f be defined on the interval $[0, 1]$ by $f(x) = 2x + 1$. Derive two different Fourier series which each give a representation of f valid on the interval $(0, 1)$ at least.

Answer:

8. Solve the heat conduction problem:

$$u_{xx} = 4u_t, \qquad 0 < x < \pi, \quad t > 0;$$
$$u(0,t) = 10, \qquad u_x(\pi, t) = 0, \quad t > 0;$$
$$u(x,0) = \sin(3x/2) + 10, \quad 0 < x < \pi.$$

Hint: Here the left end of the bar is kept at fixed temperature 10, while the right end is insulated. There is no "ready formula" for this case, you have to work from scratch.

Answer:

9. Solve the wave problem:

$$u_{xx} = u_{tt}, \qquad 0 < x < \pi, \quad t > 0;$$
$$u(0,t) = 0, \qquad u(\pi, t) = 0, \quad t \geq 0;$$
$$u(x,0) = \sin(4x) + 2\sin(6x), \quad u_t(x,0) = \sin(10x), \quad 0 \leq x \leq \pi.$$

Answer: