21-260: Differential Equations
Final Exam Formula Sheet Suggestions

1. For the Final Exam you are allowed to bring a 8.5×5.5 inches piece of paper with formulas written on both sides. This is half of a usual sheet of paper. You can write anything you want on this formula sheet. Here is what I would consider writing if I were a student in this course.

   By the way, when I omit information on this sheet, I am not implying that it will not be on the exam. I believe that the formulas below might be harder to memorize for some students.

2. What not to write: anything with the Laplace transform. The Laplace transform table will come with the exam.

3. The solution of $y' + p(t)y = g(t)$ is

$$y = \frac{1}{\mu(t)}\left(\int \mu(t)g(t)dt + C\right), \quad \text{where } \mu(t) = \exp\left(\int p(t)dt\right).$$

4. If $M(x,y)dx + N(x,y)dy = 0$ is exact (which happens when $M_y = N_x$), the solution is $\psi(x,y) = C$, where ψ is found by solving $\psi_x = M$ and $\psi_y = N$.

5. If $M(x,y)dx + N(x,y)dy = 0$ is not exact, you might be able to find and integrating factor that will make it exact. If $(M_y - N_x)/N$ is a function of x only, then there is an integrating factor $\mu(x)$ that is found by solving

$$\frac{d\mu}{dx} = \frac{M_y - N_x}{N}\mu.$$

   If $(N_x - M_y)/M$ is a function of y only, then there is an integrating factor $\mu(y)$ that is found by solving

$$\frac{d\mu}{dy} = \frac{N_x - M_y}{M}\mu.$$

6. To solve $y' = f(t,y)$, where $y(0) = 0$, using the method of successive approximations, you go like this:

$$\phi_0(t) = 0,$$

$$\phi_1(t) = \int_0^t f(s, \phi_0(s))ds,$$

$$\vdots$$

$$\phi_{n+1}(t) = \int_0^t f(s, \phi_n(s))ds.$$

Then $y(t) = \lim_{n\to\infty} \phi_n(t)$.

7. When it comes to first order linear systems with constant coefficients, there are only three things I believe one can forget:

   (a) In the case of 2×2 matrices: the types of phase portraits and the stability of the origin. You might need four tiny pictures to remember how a saddle, a (proper) node, an improper node and a spiral look like.

   (b) In the cases of 2×2 and 3×3 real matrices, if there is an eigenvalue λ repeated twice, then λ has to be real, because complex eigenvalues of real matrices come in conjugate pairs. The two fundamental solutions that correspond to λ are

   $$\bar{v}e^{\lambda t} \text{ and } (t\bar{v} + \bar{u})e^{\lambda t},$$

   where $\bar{v}$ in the eigenvector and $\bar{u}$ is the generalized eigenvector corresponding to λ. The generalized eigenvector $\bar{u}$ is found from $(A - \lambda I)\bar{u} = \bar{v}$.

   (c) Non-homogeneous systems: let $\Psi(t)$ be the fundamental matrix of $\bar{x}' = A\bar{x} + \bar{g}(t)$. Then $\bar{u}'(t) = \Psi^{-1}(t)\bar{g}(t)$. Don't forget to integrate $\bar{u}'(t)$. Then $\bar{x}(t) = \Psi(t)\bar{u}(t)$.

8. Particular solutions in second order linear differential equations with constant coefficients: some might need the table from Section 3.5. I think that once you understand the method, you don't need the table. But anyway, it's up to you.

9. The Fourier series of $f(x)$ on $[-L, L]$ is

   $$\frac{a_0}{2} + \sum_{n=1}^{\infty}[a_n \cos(\frac{n\pi x}{L}) + b_n \sin(\frac{n\pi x}{L})],$$

   where $a_0 = \frac{1}{L}\int_{-L}^{L} f(x)dx$; $a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos(\frac{n\pi x}{L})dx$; $b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin(\frac{n\pi x}{L})dx$.

10. The sine Fourier series of $f(x)$ on $[0, L]$ is

    $$\sum_{n=1}^{\infty} b_n \sin(\frac{n\pi x}{L}),$$

    where $b_n = \frac{2}{L}\int_0^L f(x)\sin(\frac{n\pi x}{L})dx$.

11. The cosine Fourier series of $f(x)$ on $[0, L]$ is

    $$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(\frac{n\pi x}{L}),$$

    where $a_0 = \frac{2}{L}\int_0^L f(x)dx$; $a_n = \frac{2}{L}\int_0^L f(x)\cos(\frac{n\pi x}{L})dx$.

12. Heat equation (bar ends kept at constant temperature):

$$\alpha^2 u_{xx} = u_t, \qquad 0 < x < L, \quad t > 0;$$
$$u(0,t) = T_1, \qquad u(L,t) = T_2, \quad t > 0;$$
$$u(x,0) = f(x), \qquad 0 < x < L.$$

Solution:

$$u(x,t) = (T_2 - T_1)\frac{x}{L} + T_1 + \sum_{n=1}^{\infty} c_n e^{-n^2\pi^2\alpha^2 t/L^2} \sin(\frac{n\pi x}{L}),$$

where c_n is the coefficient of $\sin(\frac{n\pi x}{L})$ in the sine series of $f(x) - (T_2 - T_1)\frac{x}{L} - T_1$ over $[0, L]$. In other words $c_n = \frac{2}{L}\int_0^L (f(x) - (T_2 - T_1)\frac{x}{L} - T_1)\sin(\frac{n\pi x}{L})dx$.

Notice that if you take $T_1 = T_2 = 0$, you obtain the original formula for the case when both end are kept at temperature zero. So there is no reason to waste space on the case when both ends are kept at temperature zero.

13. Heat equation (bar ends insulated):

$$\alpha^2 u_{xx} = u_t, \qquad 0 < x < L, \quad t > 0;$$
$$u_x(0,t) = 0, \qquad u_x(L,t) = 0, \quad t > 0;$$
$$u(x,0) = f(x), \qquad 0 < x < L.$$

Solution:

$$u(x,t) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n e^{-n^2\pi^2\alpha^2 t/L^2} \cos(\frac{n\pi x}{L}),$$

where c_n is the coefficient of $\cos(\frac{n\pi x}{L})$ in the cosine series of $f(x)$ over $[0, L]$. In other words $c_n = \frac{2}{L}\int_0^L f(x)\cos(\frac{n\pi x}{L})dx$ for $n \geq 0$.

14. Wave equation:

$$\alpha^2 u_{xx} = u_{tt}, \qquad 0 < x < L, \quad t > 0;$$
$$u(0,t) = 0, \qquad u(L,t) = 0, \quad t \geq 0;$$
$$u(x,0) = f(x), \quad u_t(x,0) = g(x), \quad 0 \leq x \leq L.$$

Solution:

$$u(x,t) = \sum_{n=1}^{\infty} c_n \sin(\frac{n\pi x}{L})\cos(\frac{n\pi\alpha t}{L}) + \sum_{n=1}^{\infty} k_n \sin(\frac{n\pi x}{L})\sin(\frac{n\pi\alpha t}{L})$$

Here c_n is the coefficient of $\sin(\frac{n\pi x}{L})$ in the sine series of $f(x)$ over $[0, L]$. In other words

$$c_n = \frac{2}{L}\int_0^L f(x)\sin(\frac{n\pi x}{L})dx.$$

And $\frac{n\pi\alpha}{L}k_n$ is the coefficient of $\sin(\frac{n\pi x}{L})$ in the sine series of $g(x)$ over $[0, L]$. In other words

$$k_n = \frac{2}{n\pi\alpha}\int_0^L g(x)\sin(\frac{n\pi x}{L})dx.$$

15. Here are some trigonometric formulas you won't regret:

$$\sin(\alpha)\sin(\beta) = \tfrac{1}{2}(\cos(\alpha-\beta) - \cos(\alpha+\beta))$$
$$\cos(\alpha)\cos(\beta) = \tfrac{1}{2}(\cos(\alpha-\beta) + \cos(\alpha+\beta))$$
$$\sin(\alpha)\cos(\beta) = \tfrac{1}{2}(\sin(\alpha+\beta) + \sin(\alpha-\beta))$$

16. To be honest, I would write down only formulas 3 (?), 5 (?), 12, 13, 14, 15. Everything else most can memorize. You should not expect to be able to write down EVERYTHING.