Last Name: ______________________________

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Concepts of Mathematics
Final Exam (Version A)
December 13, 2007

Please circle your section:

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Problem	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
Bonus	3	
Total	83	

1. You do not have to justify or explain your answers to the following questions.

(a) (2 pts.) Let α, β and γ be the solutions of the equation

$$x^3 - 9x^2 - 4x + 36 = 0.$$

Find $\alpha + \beta + \gamma$ without solving the equation.

$$\alpha + \beta + \gamma = 9.$$

(b) (2 pts.) Let $S = \{1, 2\}$. What is the power set $\mathcal{P}(S)$?

$$\mathcal{P}(S) = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}.$$

(c) (2 pts.) In how many ways can the letters of **ANTANANARIVO** be rearranged?

$$\frac{12!}{4!3!}.$$

(d) (2 pts.) How many congruence class solutions modulo 6 does the equation $9x \equiv 3 \pmod 6$ have?

Three solutions.

(e) (2 pts.) What is the last digit of 9^{100}?

The last digit is 1.

2. (10 pts.) Prove that for all natural numbers $n > 2$ we have that

$$3^n > 4n + 5.$$

Base Case: When $n = 3$, we have that $3^3 = 27 > 4 \cdot 3 + 5 = 17$.

Induction Hypothesis: Suppose that for some integer $k > 2$ we have that

$$3^k > 4k + 5. \ (*)$$

Want to prove: $3^{k+1} > 4(k + 1) + 5$, that is $3^{k+1} > 4k + 9$.

By multiplying both sides of $(*)$ by 3 we get $3^{k+1} > 12k + 15$. Thus it is enough to prove that $12k + 15 \geq 4k + 9$ when $k > 2$. Notice that $12k + 15 \geq 4k + 9$ is equivalent to $8k + 6 \geq 0$, which is true when $k > 2$. Thus $3^{k+1} > 12k + 15 \geq 4k + 9$ and $3^{k+1} > 4k + 9$.

Q.e.d.

3. Define $f : \mathbb{Z}^2 \to \mathbb{Z}$ by $f(x, y) = x^2 + 3y$.

(i) (5 pts.) Is f an injection? Supply a proof or counterexample.

The function f is not injective. For example, $f(2, 0) = f(1, 1)$.

(ii) (5 pts.) Is f a surjection? Supply a proof or counterexample.

The function f is not surjective. We will prove it by contradiction.

Suppose f is surjective, then there exist integers x and y such that $x^2 + 3y = 2$. So $x^2 = -3y + 2$, which means that $x^2 \equiv 2 \pmod 3$.

Notice that if $x \equiv 0 \pmod 3$, then $x^2 \equiv 0 \pmod 3$.
If $x \equiv 1 \pmod 3$, then $x^2 \equiv 1 \pmod 3$.
If $x \equiv 2 \pmod 3$, then $x^2 \equiv 1 \pmod 3$.

Thus $x^2 \equiv 2 \pmod 3$ is not possible and so f is not surjective.

4. (10 pts.) Prove the following summation formula by counting a set in two ways.

$$\sum_{i=0}^{n} \binom{i+2}{i} = \binom{n+3}{n}.$$

Be very explicit.

Let $S = \{x_1, x_2, x_3, \ldots, x_n, x_{n+1}, x_{n+2}, x_{n+3}\}$.

$\binom{n+3}{n}$ counts the number of n element subsets of S.

$\binom{i+2}{i}$ counts the number of n element subsets A of S such that the element with the largest index in $S - A$ is x_{i+3}. As i varies from 0 to n, the sum $\sum_{i=0}^{n} \binom{i+2}{i}$ counts all possible n element subsets of S.

Q.e.d.

5. (10 pts.) Suppose that the integers m, n and k satisfy the equation

$$n^2 + m^2 = k^2.$$

Prove that if k is divisible by 3, then both m and n are divisible by 3.

If $x \equiv 0 \pmod 3$, then $x^2 \equiv 0 \pmod 3$.
If $x \equiv 1 \pmod 3$, then $x^2 \equiv 1 \pmod 3$.
If $x \equiv 2 \pmod 3$, then $x^2 \equiv 1 \pmod 3$.

Thus, if $n^2 + m^2 \equiv 0 \pmod 3$, we need that both $n \equiv 0 \pmod 3$ and $m \equiv 0 \pmod 3$.

Q.e.d.

6. (10 pts.) A fair six-sided die is rolled 4 times. What is the probability that each of the values 1, 2, 3 appears during the experiment?

Hint: Use the Method of Inclusion-Exclusion.

There are many ways to solve this problem, only one is present here. Also, one didn't have to compute the exact number at the end to get full credit.

Let A_i be the event that the value i does not appear. We need to compute $P({A_1}^c \cap {A_2}^c \cap {A_3}^c)$.

$$P({A_1}^c \cap {A_2}^c \cap {A_3}^c) = 1 - P(A_1 \cup A_2 \cup A_3) =$$

$$= 1 - P(A_1) - P(A_2) - P(A_3) + P(A_1 \cap A_2) + P(A_1 \cap A_3) + P(A_2 \cap A_3) - P(A_1 \cap A_2 \cap A_3) =$$

$$= 1 - 3\frac{5^4}{6^4} + 3\frac{4^4}{6^4} - \frac{3^4}{6^4} = \frac{1}{12}.$$

7. (10 pts.) There are three closed boxes on the table. The First Box has 2 red and 3 black balls, the Second Box has 3 red and 2 black balls and the Third Box has 5 red balls. A man rolls a die. If the result is 1, he opens the First Box; if the result is even, he opens the Second Box; if the result is either 3 or 5, he opens the Third Box. After that he picks a ball from the open box. What is the probability that the other four balls in the box are red given that the selected ball is red?

$$P(\text{Box III}|\text{red}) =$$

$$= \frac{P(\text{red}|\text{Box III})P(\text{Box III})}{P(\text{red}|\text{Box I})P(\text{Box I}) + P(\text{red}|\text{Box II})P(\text{Box II}) + P(\text{red}|\text{Box III})P(\text{Box III})} =$$

$$= \frac{1 \cdot \frac{1}{3}}{\frac{2}{5} \cdot \frac{1}{6} + \frac{3}{5} \cdot \frac{1}{2} + 1 \cdot \frac{1}{3}} = \frac{10}{21}.$$

8. (10 pts.) Suppose that n different letters are sent to n different addresses on the same street, one to each address. A drunk mailman **randomly** delivers the letters to the n addresses on the street, one to each address. What is the expected number of letters that were received at correct addresses?

Label addresses by $1, 2, \ldots, n$. Let X_i be the random variable that gets value 1 if the correct letter is recieved at address i; otherwise X_i is zero.

Notice that the random variable $X = X_1 + X_2 + \ldots + X_n$ gives the number of letters received at correct addresses.

By linearity of expectation $E(X) = E(X_1) + E(X_2) + \ldots + E(X_n)$.

$$E(X_i) = 0 \cdot P(X_i = 0) + 1 \cdot P(X_i = 1) = P(X_i = 1) = 1/n.$$

Thus $E(X) = n \cdot 1/n = 1$.

Extra Credit. (3 pts.) Prove that if p is a prime number, then

$$(p-1)! \equiv -1 \pmod{p}.$$

For any natural number $k \leq p-1$, the equation $kx \equiv 1 \pmod{p}$ has only one congruence class solution modulo p because k and p are relatively prime. Thus, for any natural number $k \leq p-1$, there exists a unique natural number $k' \leq p-1$ such that $kk' \equiv 1 \pmod{p}$.

Let's find all pairs of k and k' in which $k = k'$, so $k^2 \equiv 1 \pmod{p}$. Then we have that $k^2 - 1 = (k-1)(k+1)$ is divisible by p, which implies that either $k-1$ or $k+1$ is divisible by p. Since k is a natural number such that $k \leq p-1$, we have that either $k = 1$ or $k = p-1$.

Thus all numbers in the set $\{2, 3, 4, \ldots, p-2\}$ can be partitioned into pairs whose product is 1 modulo p, which implies that $(p-2)! \equiv 1 \pmod{p}$. Since $p - 1 \equiv -1 \pmod{p}$, we have that $(p-1)! \equiv -1 \pmod{p}$.

Q.e.d.