RESEARCH STATEMENT

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My research interests are in set theory, dynamics, and universal algebra. Much of my work has centered on structures satisfying *multiplicative invariance relations*. The basic techniques used to study such structures resemble the basic techniques used to study self-similar sets and iterated function systems, with the structures playing the role of attractors. The work also has connections to the theory of paradoxical decompositions and non-amenable groups.

In my thesis, I solved the *cube problem for linear orders*, originally posed by Sierpiński in 1958. The problem is to determine whether there exists a linear order that is isomorphic to its lexicographically ordered cartesian cube but is not isomorphic to its square. The corresponding question has been answered positively for many different kinds of structures, including groups, rings, graphs, Boolean algebras, and topological spaces of various kinds. However, the answer to Sierpiński's question is negative: every linear order isomorphic to its cube is already isomorphic to its square.

Subsequently, I solved a related problem of Sierpiński's by constructing a pair of non-isomorphic linear orders that are both left-hand and right-hand divisors of one another.

Currently, I am studying *Cantor algebras* (also called *Jónsson-Tarski algebras*) and their automorphism groups. Such algebras arise naturally when considering cube problems of the kind posed by Sierpiński, but they appear in other contexts as well. I proved a representation theorem for such algebras that in many instances makes their automorphism groups easier to describe.

I am also interested in paradoxical decompositions and the actions by non-amenable groups from which they arise. A long-term goal is to understand what can be determined about a group G from a paradoxical decomposition of a set on which G acts.

1. Self-similar structures

Suppose that $(\mathfrak{C}, \times)$ is a class of structures equipped with an associative product. For a given $A \in \mathfrak{C}$, a natural problem is to determine which structures in $\mathfrak{C}$ are invariant under left multiplication by A, that is, which structures X satisfy the isomorphism $A \times X \cong X$. One of my results is that for many classes $\mathfrak{C}$ it is possible to completely characterize such structures. Roughly speaking, they can only be obtained by replacing points in the infinite product A^ω with structures from $\mathfrak{C}$, so that tail-equivalent sequences are replaced by isomorphic structures.

To state the result precisely, we need some terminology. Given a structure A and two sequences $u, v \in A^\omega$, we say u and v are *tail-equivalent*, and write $u \sim v$, if there exist finite sequences $r, s \in A^{<\omega}$ and a tail-sequence $u' \in A^\omega$ such that $u = ru'$ and $v = su'$. The tail-equivalence class of u is denoted $[u]$.

Tail-equivalence classes are formally the smallest subsets of A^ω that are invariant under left multiplication by A. For an arbitrary subset $X \subseteq A^\omega$, if we define $A \times X$ as the set $\{au : a \in A, u \in X\}$, then $A \times X = X$ if and only if X is a union of tail-equivalence classes.

This fact can be used to produce many examples of structures invariant under left multiplication by A, as follows. Suppose that for every tail-equivalence class $[u]$ we fix a structure $I_{[u]} \in \mathfrak{C}$. Let $A^\omega(I_{[u]})$ denote the "structure" obtained by replacing every point $u \in A^\omega$ with $I_{[u]}$. The underlying set of points in $A^\omega(I_{[u]})$ is $\{(u, x) : u \in A^\omega, x \in I_{[u]}\}$. Depending on context, certain extra restrictions may need to be placed on the $I_{[u]}$ in order to make $A^\omega(I_{[u]})$ a sensible structure. This "replacement" operation generalizes the usual product, since if there is a structure Y such that $I_{[u]} = Y$ for all u, then $A^\omega(I_{[u]})$ is simply $A^\omega \times Y$.

Structures of this form are naturally invariant under left multiplication by A: if $X = A^\omega(I_{[u]})$, then $A \times X \cong X$ under the isomorphism $(a, u, x) \mapsto (au, x)$. The fact that structures replacing tail-equivalent points are identical is necessary for this map to make sense.

It turns out that this is the only way to form such structures.

"Theorem". (E.) Fix a class of structures $\mathfrak{C}$ and a structure $A \in \mathfrak{C}$. For any structure X, we have $A \times X \cong X$ if and only if X is isomorphic to a structure of the form $A^\omega(I_{[u]})$.

How to turn this "theorem" into a theorem depends on the class $\mathfrak{C}$. Here are some examples:

Theorem 1. (E.)

a. Fix a set A. Then for any set X, there is a bijection between $A \times X$ and X if and only if $X \cong A^\omega(I_{[u]})$ for some collection of sets $I_{[u]}, u \in A^\omega$. This holds even in the absence of the axiom of choice.
b. Fix a group G, and suppose X is a group such that $G \times X \cong X$. Then there is a subgroup $H \leq G^\omega$ that is closed under tail-equivalence, and a normal subgroup $N \trianglelefteq X$, such that X/N is isomorphic to H.
c. Fix a topological space T. For any topological space X, we have $T \times X \cong X$ if and only if $X \cong T^\omega(I_{[u]})$, where the topology on T^ω can be the product topology, the box topology, or any intermediate topology that is "closed under multiplication by T."
d. Fix a linear order L and let $\times$ denote the lexicographical product. Then for any order X, we have $L \times X \cong X$ if and only if $X \cong L^\omega(I_{[u]})$ for some collection of linear orders $I_{[u]}$.

An *iterated function system* (IFS) is a finite collection of contraction mappings $\{f_1, \ldots, f_n\}$ on some complete metric space. A fundamental result, due to Hutchinson [7], is that any such system has a unique attractor. That is, there is a unique compact set K such that $K = \bigcup f_i(K)$. Moreover, this attractor is naturally homeomorphic to a quotient of Cantor space (on n symbols), and under this homeomorphism each f_i becomes the shift map $u \mapsto iu$.

Theorem 1 can be viewed as an analogue to Hutchinson's result. If A and X are structures such that $A \times X \cong X$, then X can be decomposed into "A-many copies of itself." Hence there is a collection of mappings $\{f_a : a \in A\}$ such that for each $a \in A$, the map f_a sends X onto the ath copy of itself within itself, and we have $X = \bigcup f_a(X)$. Moreover there is a natural isomorphism identifying X, not as a quotient of Cantor space, but as a replacement of A^ω. Under this isomorphism the f_a become shift maps on A^ω. Since there is no notion of metric, the f_a are not contractions. As a result, the iterated images of X under a sequence of these maps need not converge to a point, as they do in the case of an IFS. However, they do converge to a substructure (or, in certain instances, the "coset of a substructure"), and it is possible to show that substructures associated to tail-equivalent sequences are isomorphic.

2. Cube Problems

It is often possible that in a given class $(\mathfrak{C}, \times)$ one can find an infinite structure X that is isomorphic to its own square. If X is isomorphic to X^2, then it is also isomorphic to X^3. The question of whether the converse holds for a given class $\mathfrak{C}$, that is, whether $X^3 \cong X \implies X^2 \cong X$ for all $X \in \mathfrak{C}$, is called the *cube problem* for $\mathfrak{C}$. If it has a positive answer, then $\mathfrak{C}$ is said to have the *cube property*.

The cube problem is related to two other basic questions concerning the multiplication of structures in a given class.

1. Does $A \times Y \cong X$ and $B \times X \cong Y$ imply $X \cong Y$ for all $A, B, X, Y \in \mathfrak{C}$? Equivalently, does $A \times B \times X \cong X$ imply $B \times X \cong X$ for all $A, B, X \in \mathfrak{C}$?
2. Does $X^2 \cong Y^2$ imply $X \cong Y$ for all $X, Y \in \mathfrak{C}$?

Question 1 is called the *Schroeder-Bernstein problem*, and question 2 is called the *unique square root problem*. Taken together, these two questions are sometimes called the *Kaplansky test problems*, after Irving Kaplansky who posed them in [11] as a heuristic test for whether a given class of abelian groups has a satisfactory structure theory ("I do believe their defeat is convincing evidence that no reasonable invariants exist."). They were considered previously by Tarski [18] and Hanf [6] for the class of Boolean algebras, and subsequently have been solved for many different classes of structures.

If the cube problem for $\mathfrak{C}$ has a negative answer, that is, if there exists an $X \in \mathfrak{C}$ such that $X^3 \cong X$ but $X^2 \ncong X$, then both the Schroeder-Bernstein problem and unique square root problem have negative answers. In practice, it is often by constructing such an X that these problems are solved.

If $\mathfrak{C}$ contains no infinite structure isomorphic to its cube, then the cube property for $\mathfrak{C}$ trivially holds. When it does not hold trivially, typically the cube property fails. Early on, Hanf showed [6] that there exists a Boolean algebra isomorphic to its cube but not its square. Tarski [19] and Jónsson [9] showed the failure of the cube property for the class of groups, as well as many other classes of algebraic structures. In the years following, the cube property was shown to fail for a large number of topological, algebraic, and relational classes of structures. See [1] [2] [5] [8] [12] [13] [21] [22] [23]. My paper [3] provides a detailed list of these results and further historical context.

It has also been shown that in rare instances the cube property holds nontrivially (see [3]). However, historically in all such cases it is actually possible to establish the stronger Schroeder-Bernstein property.

In his 1958 book *Cardinal and Ordinal Numbers*, Sierpiński posed the cube problem for the class $(LO, \times_{lex})$ of linear orders with the lexicographical product (see [15], page 232). For this class, the problem is to determine whether there exists a linear order that is isomorphic to its lexicographically ordered cube but not to its square. As mentioned, the cube problem was subsequently solved for many other classes of structures, but Sierpiński's question remained open. Still, it was known already to Sierpiński that both of Kaplansky's problems have negative answers for the class of linear orders, and it seems it was expected that the cube problem also has a negative answer.

It turns out, however, that the cube problem for $(LO, \times_{lex})$ has a positive answer.

Theorem 2. (E.) If X is a linear order such that $X^3 \cong X$, then $X^2 \cong X$. More generally, for any order X and $n > 1$ we have $X^n \cong X \implies X^2 \cong X$.

Thus the cube property holds for the class of linear orders despite the fact that both the Schroeder-Bernstein property and unique square root property fail. There are even weaker statements implying the cube property that are known to fail for $(LO, \times_{lex})$—see below. In this sense, the cube property for linear orders is closer to failing than it is for other classes of structures for which it is known to hold. The proof of the theorem can be found in my paper [3]. It uses crucially the representation yielded by Theorem 1(d.).

In *Cardinal and Ordinal Numbers*, Sierpiński posed several other questions related to the cube problem concerning the multiplication of linear orders.

1. (Sierpiński) Do there exist non-isomorphic countable orders X and Y that are right-hand divisors of one another? That is, do there exist countable orders $X \ncong Y$ such that $X \cong A \times Y$ and $Y \cong B \times X$ for some orders A, B?

In other words, Sierpiński is asking for countable witnesses to the failure of the (left-sided) Schroeder-Bernstein property. He was aware of distinct *uncountable* orders that divide one another on the right. It follows from the work in my paper [3] that the uncountability is in fact necessary.

Theorem 3. (E.) If X and Y are countable orders such that divide one another on the right, then $X \cong Y$.

A more delicate question is the following:

2. (Sierpiński) Do there exist non-isomorphic orders X and Y that are both right-handed *and* left-handed divisors of one another? That is, are there orders $X \ncong Y$ such that for some A_0, B_0, A_1, B_1 we have $X \cong A_0 \times Y \cong Y \times B_0$ and $Y \cong A_1 \times X \cong X \times B_1$?

As already indicated, Sierpiński was aware of examples of non-isomorphic orders X_0, Y_0 that divide each other on the right. Separately he knew of non-isomorphic orders X_1, Y_1 that divide each other on the left. (In other words, he was aware of examples witnessing the failure of the left-sided Schroeder-Bernstein property, and separately, the right-sided Schroeder-Bernstein property.) It is natural to ask if there are distinct orders that divide each other on both sides. If there were an order X isomorphic to X^3 but not X^2, then the pair X, X^2 would give a positive answer. By Theorem 2 there are no such orders, but it turns out the answer to Sierpiński's question is still positive.

Theorem 4. (E.) There exist non-isomorphic orders X, Y of size $2^{\aleph_0}$ that divide one another on both the left and right.

See [4]. While such orders are necessarily uncountable, it is unknown if they can consistently have cardinality smaller than $2^{\aleph_0}$. The theorem gives further evidence that the cube property for $(LO, \times_{lex})$ is "close" to being false.

2.1. **Problems and directions.**

2.1.1. *Sierpiński's other problems.* Two questions from *Cardinal and Ordinal Numbers* remain unresolved.

Q1. Do there exist linear orders X, Y such that $X^3 \cong Y^3$ but $X^2 \ncong Y^2$?
Q2. Do there exist linear orders X, Y such that $X^2 \cong Y^2$ but $X^3 \ncong Y^3$?

Sierpiński was motivated to ask these questions after seeing Morel's examples of non-isomorphic orders X, Y whose squares are isomorphic [14]. It does not seem possible to adapt Morel's construction to get positive answers for either of these questions.

Both questions are related to a generalization of the cube problem. If, for any fixed $n > 2$, it were possible to find an order X isomorphic to X^n but to none of its intermediate powers X^k, $1 < k < n$, then both Questions 1 and 2 would have positive answers (in fact, it would be enough to have such orders for $n = 5$ and $n = 7$). By Theorem 2 no such orders exist, but it may still be that these questions have positive answers.

These questions, as well as those discussed in the previous section, are instances of a much more general problem. Given a class of structures $(\mathfrak{C}, \times)$ and a semigroup $(S, \cdot)$, we say that S can be *represented* in $\mathfrak{C}$ if there is a map $i : S \to \mathfrak{C}$ such that for all $a, b \in S$, we have $i(a \cdot b) \cong i(a) \times i(b)$ and $a \neq b$ implies $i(a) \ncong i(b)$. The statement that there is an $X \in \mathfrak{C}$ isomorphic to its cube but not its square is equivalent to the statement that $\mathbb{Z}_2$ can be represented in $\mathfrak{C}$. It is typical that when the cube property fails for $\mathfrak{C}$ that it is possible to prove much more spectacular representation results. For example, Ketonen showed that every countable commutative semigroup can be represented in the class $(BA, \times)$ of countable Boolean algebras under the cartesian product, and Trnková showed that every finite abelian group can be represented in the class of compact metric spaces.

Theorem 2 is equivalent to the statement that $\mathbb{Z}_n$ cannot be represented in $(LO, \times)$ for any $n > 1$.

Q3. Which semigroups can be represented in $(LO, \times)$?
Q4. Can any non-trivial group be represented in $(LO, \times)$?

A complete answer to Question 3 would yield answers to Question 1, 2, and 4. However, evidence suggests that answering Question 3 may be more difficult than answering the corresponding question for other classes of structures. Question 4 on the other hand seems much more tractable. By Theorem 2, if the answer to Q4 is positive, any non-identity element in the witnessing group must have infinite order. I am interested in working on all of these problems.

2.1.2. *Algebra in an arbitrary class* $(\mathfrak{C}, \times)$. My approach to the cube problem for the class of linear orders is original in that it uses Theorem 1(d.) as a starting point. Often, solving the cube problem for a given class $(\mathfrak{C}, \times)$ requires an ad hoc construction, especially when the solution is negative. But Theorem 1 applies to many different classes of structures, and it would be interesting to know if it can be used to find a "universal construction" that solves the cube problem simultaneously for these various classes.

Q5. Can Theorem 1 be used to solve the cube problem for other classes of structures besides $(LO, \times)$?

3. Cantor algebras

Let X be an infinite set, or a singleton. A *Cantor algebra* is a system $\langle X, *\rangle$, where $* : X \times X \to X$ is a bijection. We think of $*$ as a product on X. Then, because $*$ is a bijection, every $x \in X$ has a unique factorization under this product $x = x_0 * x_1$. Such algebras were originally considered by Tarski and Jónnson in [10]. Later, Smirnov proved some fundamental facts about free Cantor algebras and their automorphism groups [16] [17]. Automorphism groups of Cantor algebras have arisen in other contexts. For example, the well-known Thompson group F appears as a certain subgroup of the automorphism group of the free Cantor algebra on one generator.

The algebraic structure of a Cantor algebra on even a single generator can be complicated, and in studying these algebras one seeks examples that can be readily visualized. Such examples are supplied by *block algebras*, which we now define.

Let C be any nonempty set, which we think of as a set of colors. Let B be any set of functions of the form $f : 2^\omega \to C$. We say that B is closed under dyadic concatenation if whenever $f, g \in B$, there is a function $h \in B$ that is "f on the left and g on the right." That is, for every sequence of the form $0u \in 2^\omega$ with a leading 0, we have $h(0u) = f(u)$, and for every sequence of the form $1u$ we have $h(1u) = g(u)$. We write $h = f \wedge g$. We say B is closed under dyadic division if whenever $f \in B$, there are functions $f_0, f_1 \in B$ that are the "left and right parts" of f respectively. That is, $f_0(u) = f(0u)$ and $f_1(u) = f(1u)$ for every $u \in 2^\omega$. From this it follows that $f = f_0 \wedge f_1$. If B is closed under both dyadic concatenation and division, we say that $\langle B, \wedge\rangle$ is a block algebra. We think of the functions in B as colorings of 2^ω ("blocks") that can be split in half to form new blocks, and also reassembled half-by-half to form new blocks.

It is immediate that in any block algebra B, the concatenation operator $\wedge$ is a bijection of $B \times B$ with B. Hence any block algebra is a Cantor algebra. Algebraic properties of elements in a block algebra, and of the algebra itself, are reflected in how the elements are colored. For example, idempotent elements (that is, blocks that split into two copies of themselves) appear as blocks of a single color. Viewing B as a Cantor algebra, these are the elements x that factor as $x * x$.

I showed that in fact every Cantor algebra can be realized as a block algebra.

Theorem 5. (E.) Suppose that $\langle X, *\rangle$ is a Cantor algebra. Then there is a block algebra $\langle B, \wedge\rangle$, on some collection of colors C, that is isomorphic to X.

Hence, in studying Cantor algebras one may deal only with the more visually available block algebras without any loss of generality.

From this representation theorem it is easy to give short proofs of some of the basic facts about Cantor algebras, originally established by longer algebraic means.

Theorem. (Jónnson-Tarski) Any two free Cantor algebras on finitely many generators are isomorphic.

Theorem. (Smirnov) Any countably generated Cantor algebra can be embedded in an algebra with a single generator.

Theorem. (Smirnov) There is a family of $2^{\aleph_0}$ pairwise non-isomorphic rigid Cantor algebras, each on a single generator. Hence, there is no universal countable Cantor algebra.

3.1. **Problems and directions.**

3.1.1. *Morphisms of Cantor algebras.* Another advantage of viewing Cantor algebras as block algebras is that morphisms of the algebra can be induced by maps between sets of colors. If B is a block algebra on a set of colors C, and C' is another set of colors, then any map $i : C \to C'$ induces a homomorphism $h : B \to B'$, where B' is the block algebra obtained by recoloring the blocks in B according to the map i. That is, for $f \in B$, we have $h(f) : 2^\omega \to C'$ is the block colored by the rule $h(f)(u) = i(f(u))$, and B' is simply $\{h(f) : f \in B\}$. If i is injective, then h is an isomorphism. If moreover $C' = C$ and B is closed under the recoloring i, then h is an automorphism of the original algebra B.

Let us call morphisms induced by maps on colors *color morphisms*. Somewhat surprisingly, if X is a Cantor algebra and B is the representation of X as a block algebra obtained by the proof of Theorem 5, then *all* morphisms on B are color morphisms. However, in this representation the blocks of B are colored in a maximally complicated way (in a precise sense), and often the algebra X can be represented as a block algebra on significantly fewer colors. I would like to know if there are canonical representations of Cantor algebras as block algebras where the number of colors used is minimal.

Q6. Given a Cantor algebra X, is there a canonical representation of X as a block algebra which uses the minimal number of colors? Is there such a representation in which all automorphisms are color automorphisms?

3.1.2. *Characterizing the isomorphism $X^2 \cong X$.* Theorem 5 can be viewed as an analogue to Theorem 1(a.) where the role of the bijection $f : A \times X \to X$ is now played by the bijection $* : X \times X \to X$. It is reasonable to hope that, using Theorem 5, one could characterize, within various classes of structures, those structures that satisfy the isomorphism $X^2 \cong X$, in analogy with Theorem 1(b.), (c.), and (d.).

Q7. Is it possible to characterize, for an arbitrary class of structures $(\mathfrak{C}, \times)$, the structures $X \in \mathfrak{C}$ satisfying the isomorphism $X^2 \cong X$? More generally, for a fixed $n > 1$, is it possible to characterize the isomorphism $X^n \cong X$?

4. Paradoxicality and amenability

Suppose that G is a group and X is a set on which G acts. We say that X admits a *G-paradoxical decomposition* if there is a partition of X as $X = A_1 \cup \ldots \cup A_n \cup B_1 \cup \ldots \cup B_m$ and a collection of group elements $g_1, \ldots, g_n$, $h_1, \ldots, h_m$ in G such that

$$X = \bigcup_i g_i A_i = \bigcup_j h_j B_j.$$

A group G is said to be *paradoxical* if its action on some set X yields a G-paradoxical decomposition of X. Equivalently, G is paradoxical if it acts paradoxically on itself by left multiplication. Tarski showed that a group is paradoxical if and only if it is non-amenable.

If a group G acts paradoxically on X, then in some sense X can be split into two copies of itself according to the action of G. The condition that $X \cong 2 \times X$ also says that in some sense X can be split into two copies of itself, and this latter relation is characterized by Theorem 1. While these two senses of "X can be split into two copies of itself" are different, the proof of Theorem 1 is flexible, and yields a partial "representation theorem" for a G-paradoxical decomposition of a set X. While this representation gives a lot of information about the G-paradoxical decomposition of X, such a decomposition gives only partial information about the group G from which it arises. I am interested in determining to what extent it is possible to construct a group G only from the data of G-paradoxical decomposition of a set X.

Q8. Suppose that X is an infinite set, and we have a partition $X = A_1 \cup \ldots \cup A_n \cup B_1 \cup \ldots \cup B_m$. Suppose that $f_1, \ldots, f_n, f'_1, \ldots, f'_m$ are partial injections on X such that $X = \bigcup_i f_i A_i = \bigcup_j f'_j B_j$. Can the proof of Theorem 1 be used to effectively construct a group of permutations G of X that contains elements $g_1, \ldots, g_n$ extending $f_1, \ldots, f_n$ and $h_1, \ldots, h_m$ extending $f'_1, \ldots, f'_m$?

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