Some of HW8

**Notation:** $A \cong B$ means $A$ is isomorphic to $B$.


Prove there is a countable family $F$ of countable structures, such that every countable structure in this language is isomorphic to a structure in the family.

Prove also that the structures in the family $F$ are pairwise non-isomorphic.

**Proof:** For $n \in \omega$, define $A(n,\infty)$ to be the structure $|A(n,\infty)|, S^{A(n,\infty)} = (\omega, \{0,1,\ldots,n-1\})$

For $m \in \omega$ define $A(\infty,m)$ to be the structure $|A(\infty,m)|, S^{A(\infty,m)} = (\omega, \{m,m+1,\ldots\})$

Define $A(\infty,\infty) = (\omega, \{0,2,4,\ldots\})$.

**Claim 1:** If $A$ is a countably infinite structure in this language, then there are $x,y \in \{0,1,\ldots,\infty\}$ such that $A$ is isomorphic to $A(x,y)$.

**Proof:** There are three possibilities: $S^A$ is finite, $|A| - S^A$ is finite, both $A$ and $S^A$ are infinite.

Suppose we are in case 1, and $S^A$ is of size $n$. Choose an enumeration $|A| = \{a_0,a_1,\ldots,a_{n-1},a_n,\ldots\}$ so that $S^A = \{a_0,\ldots,a_{n-1}\}$ Define $\pi : A(n,\infty) \to A$ by $\pi(i) = a_i$.

Then $\pi$ is an isomorphism since it is a bijection and $i \in S^{A(n,\infty)}$ if $i \in \{0,\ldots,n-1\}$ iff $a_i \in \{a_0,\ldots,a_{n-1}\}$ if $\pi(i) \in \{\pi(a_0),\ldots,\pi(a_{n-1})\}$ if $\pi(i) \in S^A$.

Similarly for the other cases.

**Claim 2:** The structures in $F$ are pairwise non-isomorphic.

**Proof:** Fix $A(x,y)$ and $A(x',y')$ in our family s.t. $(x,y) \neq (x',y')$. (at least one of $x,y$ is $\infty$ and at least one of $x',y'$ is $\infty$.)

WLOG $x \neq x'$ and $x < x'$. Hence $x$ is finite, say $x = n$. Then $S^{A(x,y)} = \{0,\ldots,n-1\}$.

Let $\pi : |A(x',y')| \to |A(x,y)|$ be any bijection. We know $S^{A(x,y)} = \{0,\ldots,n-1\}$ And $S^{A(x',y')} = \{0,\ldots,n-1,n,\ldots\}$ is of size $x' > n$ ($x'$ possibly infinite).

Hence $\{\pi(0),\ldots,\pi(n-1),\pi(n),\ldots\}$ is of size $x'$ as well. Thus there must be some $N$ such that $N \in S^{A(x',y')}$ but $N \not\in S^{A(x,y)}$. Hence $\pi$ is not an isomorphism. Since $\pi$ was arbitrary, there is no isomorphism.

3.19b Consider the language with a single binary relation symbol $R$. Construct a family of countably many pairwise non-isomorphic countable structures in this language.

**Proof:** First, an example.

A useful way to think about isomorphisms is: if $A,B$ structures and $\pi : |A| \to |B|$ a bijection then $\pi$ is an isomorphism if when you “apply $\pi$” to $c^A,R^A,f^A$ for all the symbols in your language you get $c^B,R^B,f^B$.

Consider the structures in this language $A = (|A|, R^A) = (\{(1,2,3),(1,2),(1,3)\}, B = (|B|, R^B) = (\{(1,2,3),(2,3),(2,1)\}) C = (|C|, R^C) = (\{(1,2,3),(1,1),(2,2)\})$.

Then $A$ is isomorphic to $B$. Bijection is given by $\pi(1) = 2, \pi(2) = 3, \pi(3) = 1$; is an isomorphism because when you apply $\pi$ to $R^A = \{(1,2),(2,3)\}$ you get $\{(2,3),(3,1)\} = R^B$.

But $A$ is not isomorphic to $C$. For any bijection $\pi$ from $\{1,2,3\}$ we have “$\pi[R^A]$” = $\{(\pi(1),\pi(2)),(\pi(1),\pi(3))\}$ $\neq R^C$.

Now we prove the problem.

For every infinite $X \subseteq \omega$, we list $X$ in increasing order: $X = \{n_0,n_1,\ldots\}$

There are uncountably many infinite subsets of $\omega$.

For every such $X$, we define a relation
\[ R^A_X = \{(0,0), (0,1), \ldots, (0,n_0-1), (1,0), (1,1), \ldots, (1,n_1-1), \ldots\} \]

The point: for every \( k \in \omega \), there are exactly \( n_k \) many tuples of the form \((k, \cdot)\) in the relation.

Notice: if \( k < l \) then than number of tuples \((k, \cdot)\) is \( n_k \) which is less than \( n_l \) which is the number of tuples of the form \((l, \cdot)\).

E.g. if \( X = \{2, 4, 6, \ldots\} \) Then \( R^A_X = \{(0,0), (0,1), (1,0), (1,1), (1,2), (1,3), \ldots\} \)

We now define a structure \( A_X \) with \(|A_X| = \omega\) and \( R^A_X \) as just defined.

Claim: if \( X \neq Y \) then \( A_X \) is not isomorphic to \( A_Y \).

Proof: We write \( X = \{n_0, n_1, \ldots\}, Y = \{m_0, m_1, \ldots\} \) in increasing order.

Wlog there is \( n \in A_X \) such that \( n \notin A_Y \). Then \( n = n_k \) for some \( k \). Hence the number of tuples of the form \((k, \cdot)\) in \( R^A_X \) is \( n_k = n \).

If there were an isomorphism \( \pi : A \to B \) we would have to have that number of tuples of the form \((\pi(k), \cdot)\) in \( R^A_Y \) is also.

But since \( n \notin Y \), for every \( k \) we have that the number of tuples of the form \((k, \cdot)\) in \( R^A_Y \) is \( m_k \neq n \)

Hence there is no isomorphism, i.e. \( A_X \) and \( A_Y \) not isomorphic.
Interlude: More on structures + isomorphism.

Graph - consider long w/ single binary relation symbol R.
A graph is a structure A satisfying the following theory E:

\[ E = \{ \forall u \forall v \ (R(u,v) \land \forall u \forall v \ (R(u,v) \Rightarrow R(v,u)) \} \]

We say a graph is a set equipped w/ an irreflexive, symmetric relation.

- e.g. \( A = (\{1,2,3\}, R^A) \)
  \[ = (\{1,2,3\}, \{(1,2), (2,1), (3,1)\}) \]

is a graph.

Pic: \[ \begin{array}{c}
\text{draw an edge between } a, b \\
\text{iff } (a,b) \in R^A \\
\end{array} \]

No say \( x, y \) are adjacent in a graph \( A \) iff \( (x,y) \in R^A \).
Another graph:

\[ R^A' = \{(4,3), (3,1)\} \]

Another:

\[ R^{A''} = \{(1,1), (2,1), (2,3), (3,3)\} \]

Observe: \( A \equiv A'' \) but \( A \not\equiv A' \) (why?)

Consider \( B = (\{1,0,1\}, R^B) = (\{1,2,3\}, \{(1,2), (2,1), (1,3)\}) \)

Then \( B \) is a structure in this long, but \( B \) is not a graph. (\( R^B \) not symmetric)

\[ \text{CutCut: } -C = (\{1,1\}, R^C) = (\{1,2\}, \{(1,1), (2,1)\}) \]

Is \( C \) a graph? In fact, \( C \) is a substructure of \( A \) since \( 1c1 \subseteq 1A1 \)

\[ R^C = R^A \cap 1c1 \]
Note: $A'$ above is not a substructure of $A$ since $|A'| = |A|$ but $R^{A'} \neq R^A$.

Claim: There is an uncountable family $F$ of pairwise non-isomorphic odd infinitely infinite graphs.

Pf: For each $n \geq 1$, an $n$-star $\ast_n$ graph that looks like this:

![Graph](image)

The point: in an $n$-star:
- Center is adjacent to $n$ points.
- All other points adjacent only to center.

E.g., $A = (|A|, R^A)$

\[
A = (|A|, R^A) = (\{1,2,3,4\}, \{(1,2), (2,1), (1,3), (3,1), (1,4), (4,1)\})
\]

is a 3-star:

![Graph](image)
(iv) \[ \mathbb{P} \mathbb{S} \mathbb{O} \mathbb{S} \mathbb{X} = \{ n_0, n_1, \ldots \} \leq \mathbb{W} \cup \mathbb{N} \text{ is an infinite subset of } \mathbb{W} \text{ not containing } 0. \]

Let \( A_X \) be the graph consisting of infinitely many stars, one for each \( n \in X. \)

e.g. if \( X = \{ 2, 4, 6, \ldots \} \), \( A_X \) looks like:

\[
\begin{array}{c}
\cdots \quad 3 \\
1 \quad X \\
\cdots \quad 5
\end{array}
\]

Actually defining \( A \), explicitly isn't so important, could do:

\[
A_X = (\mathbb{W}, \{(x, x), (x, y), (3, 4), (4, 3), (3, 5), (5, 3), (3, 6), (6, 3), (3, 7), (7, 3)\})
\]

Claim: if \( X \neq Y \) then \( A_X \) and \( A_Y \) are not isomorphic.

PF: Wlog there is \( n \in X \) s.t. \( n \not\in Y \).
Hence there is \( x \in A \) such that \( u \) is adjacent to exactly \( n \)-many points (Center of \( n \)-many elements).

For any \( y \in A \setminus \{u\} \), if \( y \) is adjacent to \( m \) points for some \( m \leq n-1 \) or adjacent to exactly \( 1 \).

Hence \( A_x \not\sim A_y \)

Hence \( \exists A_x : x \in \{1, 2, \ldots, \infty\} \) is as desired.

**Linear orders**

A linear order is any structure \( A \) satisfying the theory:

\[
\forall u \left( \neg R(u, u) \right) \\
\forall u \forall v \left( R(u, v) \implies \neg R(v, u) \right) \\
\forall u \forall v \forall w \left( R(u, v) \land R(v, w) \implies R(u, w) \right)
\]

(L.O.'s are irreflexive, antisymmetric, transitive relations)

This defines a partially ordered set (P.O.S).

E.g., \((R, \leq)\) is a linear order but \((R, \leq)\) is not.
(vi)

Visualize as L.0. by drawing points in a line: if \((a,b) \in \mathbb{R}\)
then a left of b. No edges.

-\(e.g. \mathcal{A} = (\mathcal{A}, R^\mathcal{A}) = \{(1,2,3), \{(1,2), (2,3), (1,3)\}\}\)

\(w \in L.0.
\)

\(\mathcal{A}:
\)

\[
\begin{array}{c}
0 \\
1 \\
2 \\
3
\end{array}
\]

-Now consider \(\mathcal{A} = (\omega, <)\)

\(\mathcal{A}:
\)

\[
\begin{array}{ccccccc}
0 & 1 & 2 & 3 & \ldots
\end{array}
\]

-And \(\mathcal{B} = (\mathbb{Z}, <)\)

\[
\begin{array}{ccccccc}
-1 & 0 & 1 & 2 & 3 & \ldots
\end{array}
\]

Then \(|\mathcal{A}| \leq |\mathcal{B}|\) and \(R^\mathcal{A} = R^\mathcal{B} \cap (\mathcal{A})\)

so \(\mathcal{A} \cup \mathcal{A} = \text{substructure of } \mathcal{B}\)
(vii)

- IS A an elementary substructure of B? 

No: \( A = \exists u \forall v (\neg R(u, v)) \)

- Consider \( C = (\mathbb{N}, R^C) \) where

\[
\begin{align*}
\mathbb{N} &= \omega \cup \{x\} \\
R^C &= R^A \cup \{(n, x) : x \in u\}
\end{align*}
\]

Picture:

```
0 1 2 3 ...
```

- Then A is a substructure in C

- IS A an elementary in C?

No again:

\( A = \forall u \exists v (R(u, v)) \)

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elements have successors
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Since \( x \) has no successor.
What about \( C' \):

\[
\begin{array}{cccccccc}
0 & 1 & 2 & x_0 & x_1 \\
\end{array}
\]

Is \( A \) elementary in \( C' \)?

Still no:

\[
A = \exists u \forall v \left( u \neq v \Rightarrow R(u, v) \wedge \exists w \left( R(w, v) \wedge \forall z \left( R(z, w) \Rightarrow R(z, u) \right) \right) \right)
\]

"\( A \) has a unique minimal element. Every other element has a unique predecessor."

\( C' \neq \) \( x_0 \) has no predecessor

However we will prove (later):

Then, there is a linear order \( B \) s.t.

- \( A = \langle B, < \rangle \) is a linear substructure of \( B \)
- \( \exists x \in \langle B, < \rangle \) s.t. for all \( n < x \)
(iv)

- For new something weaker
- For every new let \( c_n \) be a new constant
- Let \( A^* \) be expansion of \( A = (\omega, c) \) that interprets \( c_n = n \)
- Let \( T^* \) be the set of all sentences \( \phi \) s.t. \( A^* \models \phi \)

E.g., \( T^* \) contains axioms for a linear order, indeed all sentences true in \( A \), as well as the following:

- \( \forall u \left( u \notin c_0 \right) \)
- Sentences of the form

\[
\forall n < c_{n+1} \forall u \left( c_n < u \land u < c_{n+1} \right)
\]

Hence any model \( B \models T^* \) is a linear order that looks like \( \omega \) at the beginning.

Question: If \( B \models T^* \) is \( B \) isomorphic to \( A^* \)?
We prove no!

Let C be a new constant symbol.

Let E be the set of sentences of the form $C_n < C$.

We prove $T^* \cup E$ is satisfiable.

Let $D \subseteq T^* \cup E$ be finite

$= D_0 \cup D_1$

where $D_0 \subseteq T^*$

$D_1 \subseteq E$

Let $C_{n_0}, ..., C_{n_k}$ be set of C's appearing in $D_1$

Let $N = n_k + 1$

Let $A'$ be the expansion of $A^*$ that interprets E as N.

Then $A' = D_0$ because actually $A' = T^*$

and $A' = D_1$ since $C^N$ really is larger than $C_{n}^N = n$ for all $C$'s appearing in $A$.

Hence by compactness $T^* \cup E$ is satisfiable.
Let $B$ be a model. Then:

$$B \cong \mathbb{C}^0 \cong \mathbb{C}_1 \cong \mathbb{C}_2 \cong \mathbb{C}_3 \cong \mathbb{C}_4$$

Since $B = \mathbb{C}^*$ we knew every element in $|B|$ except $c_0$ has a unique successor and predecessor.

So, $\cdots w \vdash \mathbb{C}^0$ \\
$c_0$ \vdash \mathbb{C}^1$ \\
$c_1$ \vdash \mathbb{C}^2$ \\
$c_2$ \vdash \mathbb{C}^3$ \\
$c_3$ \vdash \mathbb{C}^4$ \\
$c_4$ \vdash \mathbb{C}^5$ \\
$c_5$ \vdash \mathbb{C}^6$ \\
$c_6$ \vdash \mathbb{C}^7$ \\
$c_7$ \vdash \mathbb{C}^8$ \\
$c_8$ \vdash \mathbb{C}^9$ \\
$c_9$ \vdash \mathbb{C}^{10}$ \\
$c_{10}$ \vdash \mathbb{C}^{11}$ \\
$c_{11}$ \vdash \mathbb{C}^{12}$ \\
$c_{12}$ \vdash \mathbb{C}^{13}$