Ch. 2: Propositional Logic (PL)

- Logical system in which statements are formed from propositional variables and connectives, e.g.

\[
P_1 \land P_2 \\
P_0 \iff (P_1 \lor P_2)
\]

(think of each \(P_i\) as T/F)

- Not able to refer to actual mathematical objects (numbers, functions, relations, etc.). For this need first-order logic (FOL).

- PL still useful as warmup for FOL: in FOL the variables \(P_i\) replaced by terms:

  e.g. \((x > 0) \iff [(x > 0) \lor (x = 0)]\)

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- Need to define the syntax and semantics of PL

  - Symbols we can use
  - Which sequences of symbols are sentences
  - Which sequences of sentences are proofs

  \(\uparrow\) what it means for a sentence to be true given an interpretation of the symbols
Symbols in PL

T, I constants

\( \top, \bot, =, \in \) connectives

( ) non-logical symbol

P₀, P₁, P₂, ... propositional variables (or parameters)

Let \( A = \) set of all symbols

= \{ \top, \bot, =, \in, ( ), P₀, P₁, \ldots \}

- A string is an ordered sequence of symbols, e.g., \( P₀ \) → \( \top \top P₁ \)

- Let \( \mathcal{S} \) denote the set of all strings (really, think of \( \mathcal{S} \) as \( A^* \))

- We recursively define a special subset of \( \mathcal{S} \) called the set of sentences
* \( T \) and \( \bot \) are sentences
* \( P_0, P_1, \ldots \) are sentences
* If \( \varphi \) and \( \psi \) are sentences then 
  \[ \varphi, \psi \]
  \[ (\land, \varphi, \psi) \]
  \[ (\lor, \varphi, \psi) \]
  \[ (\neg, \varphi) \]
  \[ (\to, \varphi, \psi) \]
  \[ (\leftrightarrow, \varphi, \psi) \]
  \[ \text{conjunction of } \varphi, \psi \]
  \[ \text{disjunction of } \varphi, \psi \]
  \[ \text{negation of } \varphi \]
  \[ \text{conjunction of } \varphi, \psi \]
  \[ \text{conditional} \]
  \[ \text{bi-conditional} \]

For example:

\[ (T, P_1) \]
\[ (\lor, (\to, P_0, P_1), T) \]

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* More formally:

Define \( s: w \to P(\varphi) \) recursively:

* \( s(0) = \{ \bot, T, P_0, P_1, \ldots \} \)
* \( s(n+1) = s(n) \cup \{(\land, \varphi, \psi) \mid \varphi, \psi \in s(n)\} \)
  \[ \cup \{(\lor, \varphi, \psi) \mid \varphi, \psi \in s(n)\} \]
  \[ \cup \{(\neg, \varphi) \mid \varphi \in s(n)\} \]
  \[ \cup \{(\to, \varphi, \psi) \mid \varphi, \psi \in s(n)\} \]
  \[ \cup \{(\leftrightarrow, \varphi, \psi) \mid \varphi, \psi \in s(n)\} \]
say a string \( w \) is a sentence if there is a string \( S(w) \).

e.g. \( P_0 \in S(0) \\
\left( v, \rightarrow, P_0, P_1, \rightarrow \right) \in S(2) \\

are sentences.

Then two definitions of sentence are (clearly) equivalent.

Can you see how to define \( S \) using recursion theorem?

**Important**  
Sentences are uniquely readable: i.e.

if \( \epsilon, \phi, \chi, \Psi \) are sentences then

- \( \left( \lor, \epsilon, \phi \right) = \left( \lor, \chi, \Psi \right) \) implies \( \epsilon = \chi \)
- \( \left( v, \epsilon, \phi \right) = \left( v, \chi, \Psi \right) \) implies \( \epsilon = \chi \) \( \phi = \Psi \)

above remains true if \( v \) replaced by \( \land, \rightarrow, \rightarrow \).
- e.g. only one way to decompose $(\land, P_0, (\land, P_1))$ as a conjunction of two sentences: $P_0$ and $P_1$.

- conjuncts of arbitrary strings not uniquely readable.
  e.g. could also decompose $(\land, P_0, (\land, P_1))$
  as the "conjunct " $P_0 \land P_1$
  but there are not sentences.

- if we were honest, could prove unique readability, but we won't
- still important will be making definitions that depend on unique readability

For example: can define complexity of a sentence $\phi$ recursively:

- $I$, $T$, $P_0$, $P_1$, $\bot$ have complexity 0

- If $\phi$ has complexity $n$, then $(\land, \phi)$ has complexity $n+1$

- If $\phi$ has complexity $n$ and
  \()$ has complexity $n^*$
  then
  \((\land, \phi, x)\) has complexity $\max(n, n^*) + 1$

\(\land, x, =)\)
e.g. \((1, (7, P_1), P_2)\) has comp 2.

(Note: defn of comp. depends on unique readability).

So we'll almost always write sentences using traditional notation. i.e.

\[
\begin{align*}
&\neg \Leftrightarrow \\
&\Leftrightarrow \\
&\vee \\
&\leftrightarrow
\end{align*}
\]

More readable but be careful to insert parentheses:

\[
(\neg (P_1 \lor P_2) \lor P_3)
\]

is the same as

\[
(P_1 \lor (P_2 \lor P_3))
\]

you still have unique readability + notion of complexity for similar in traditional notation.
Induction on the constructor of Sentences

Theorem: Let \( R(x) \) be a statement about the sentence \( x \).

If \( (1) R(\varphi), R(\gamma), R(p), R(p^*), \ldots \) all hold,

\( (2) \) whenever \( R(\varphi) \) and \( R(\chi) \)
both hold then \( R(\varphi \gamma) \)
\( R(\varphi \chi) \), \( R(\varphi \Rightarrow \chi) \)
\( R(\varphi \Leftrightarrow \chi) \) all hold,

then \( R(\varphi \chi) \) holds for all sentences \( x \).

PF: Induct on the complexity of \( x 
\)
(You try)

Example: For a string \( x \),
\( \text{depth}(x) = \# \text{ of left parentheses in } x \),
\( r(x) = \# \text{ of right parentheses} \)

Prep. In any sentence \( x \), \( \text{left}(x) = \text{right}(x) \)

PF: Induct on \( \text{left}(x) = \text{right}(x) \)
(BC) true for $T, \top, P_0, P_i, \ldots$ 
(no parentheses: $l = r = c'$)

(IH) Fix $\nu, \gamma$ and assume

$$l(\nu) = r(\nu) \quad \text{and} \quad l(\gamma) = r(\gamma)$$

then

$$l((\nu \gamma)) = l(\gamma) + 1$$

$$r((\nu \gamma)) = r(\gamma) + 1$$

$$\therefore = r((\nu \gamma))$$

and

$$l((\nu \gamma)) = l(\nu) + l(\gamma) + 1$$

$$r((\nu \gamma)) = r(\nu) + r(\gamma) + 1$$

$$\therefore = r((\nu \gamma))$$

Similarly for $\lor, \Rightarrow$

never by induction $l(\nu) = r(\nu)$ for all sentences $\nu$. \checkmark

**Theorem** the set of sentences is countable (why?)

**PF:** Let $A = \{ \top, T, \lor, \land, \Rightarrow, \forall, \exists, P_0, P_1, \ldots \}$

there is a bijection $F: A \rightarrow \omega$
Namely,
\[ f(1) = 0 \]
\[ f(2) = 1 \]
\[ f(v) = 2 \]

- hence \( A \cup \text{chol} \)
- hence, since \( \omega \cup \text{chol} \)
- \( A \cup \omega \cup \text{chol} \) (why?)
- can think of \( A \cup \omega \) as the set of

\[ S \]

- let \( S \) denote the set of \( S \)
- hence \( S \subset \text{chol} \)

\[ \]
Semantics of PL

So far: sentences just particular sequences of symbols.

Now: say what it means for sentence to be T/F (relative to
truth of its parameters Pn)

- a structure is a function
  \( A : [P_n \mid n \in 3] \rightarrow [0, 1] \)

- given a structure \( A \), can extend
  to a function \( \text{Truth}_A : [e \mid e \text{ a sentence}] \rightarrow [0, 1] \) recursively, as follows:

  - \( \text{Truth}_A(T) = 1 \), \( \text{Truth}_A(L) = 0 \)
  - \( \text{Truth}_A(P_n) = A(P_n) \)
  - \( \text{Truth}_A(\neg e) = 1 \) iff \( \text{Truth}_A(e) = 0 \)
  - \( \text{Truth}_A(e \lor \gamma) = 1 \) iff \( \text{Truth}_A(e) = 1 \) or \( \text{Truth}_A(\gamma) = 1 \)
  - \( \text{Truth}_A(e \land \gamma) = 1 \) iff \( \text{Truth}_A(e) = 1 \) and \( \text{Truth}_A(\gamma) = 1 \)
  - \( \text{Truth}_A(e \Rightarrow \gamma) = 1 \) iff \( \text{Truth}_A(e) = 0 \) or \( \text{Truth}_A(\gamma) = 1 \)
  - \( \text{Truth}_A(e \Leftrightarrow \gamma) = 1 \) iff \( \text{Truth}_A(e) = \text{Truth}_A(\gamma) = 1 \)
(Note: defn depends on unique readability)

**Ex:** Define a structure $A : [P_n] \rightarrow \{0,1\}$ by $A(P_n) = 1$ if win
$= 0$ if odd

- Let $\varphi_1$ be $\neg P_1$
- $\varphi_2$ be $(P_1 \Rightarrow P_2) \land P_3$
- $\varphi_3$ be $P_1 \iff P_3$

- We have $A(P_1) = A(P_3) = 0$
$A(P_2) = 1$

- So $\text{Truth}_A(\varphi_1) = 1$
$\text{Truth}_A(\varphi_2) = 0$
$\text{Truth}_A(\varphi_3) = 1$

**Semantic consequence ($\vdash$)**
and **equivalence ($\iff$)**

- a **theory** $\Sigma$ is a set of sentences

- e.g. $\Sigma_1 = \{\varphi_1, \varphi_2, \varphi_3\}$
$= \{\neg P_1, (P_1 \Rightarrow P_2) \land P_3, P_1 \iff P_3\}$
and $\Sigma_2 = \{e_1, e_3\}$

$\{\neg P_1, P_1 \rightarrow P_3\}$

are theories.

- A structure $A$ is a model of $\varphi$ if $\text{Truth}_A(\varphi) = 1$.
  
  So we write $A \models \varphi$.

- E.g., for $A$ above we have:
  
  $A \models \neg P_1$

  $A \not\models (P_1 \rightarrow P_2) \land P_3$

- A structure $A$ is a model for a theory $\Sigma$ if $A \models \varphi$ for every $\varphi \in \Sigma$.

- E.g., for $A$ above we have:
  
  $A \models \Sigma_2$

  $A \not\models \Sigma_1$

- For sentences $\varphi, \psi$ we write $\varphi \models \psi$ ("$\varphi$ entails $\psi$") or $\psi$ is a semantic consequence of $\varphi$ if every model of $\varphi$ is a model of $\psi$. 
Let \( \varphi \) be \( P_1 \iff P_2 \).

Then \( \varphi \) is true.

Why? To prove this, use a truth table:

<table>
<thead>
<tr>
<th>( P_1 )</th>
<th>( P_2 )</th>
<th>( P_1 \iff P_2 )</th>
<th>( \varphi )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
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<td>0</td>
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<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

Notice: any assignment of truth values to \( (P_1, P_2) \) that makes \( \varphi \) true, also makes \( \varphi \) true. I.e., if \( A \) is a structure and

\[
\text{Truth}_A(P_1 \iff P_2) = 1 \quad \text{then} \quad A(P_1) = A(P_2) = 0 \\
\quad \text{or} \quad A(P_1) = A(P_2) = 1
\]

In either case \( \text{Truth}_A(P_1 \implies P_2) = 1 \)

hence \( \varphi \) is true.

If \( \Sigma \) is a theory and \( \varphi \) is a sentence we write

\[
\Sigma \models \varphi
\]

If every model of \( \Sigma \) is a model
- e.g., if \( \Sigma = \{ P_1, P_1 \Rightarrow P_2, \neg P_2 \} \)
  then \( \Sigma \vdash P_2 \)

  \[ \text{Why: if } A \vdash \Sigma \text{ then } A(P_1) = 1 \]
  \[ \Rightarrow \text{ Truth}_A(P_1 \Rightarrow P_2) = 1 \]
  \[ \text{thus } \text{ Truth}_A(P_2) = 1 \]
  \[ \therefore A \vdash P_2 \]

- \( \Sigma \) and \( \Psi \) are equivalent if \( \Sigma \vdash \Psi \) and \( \Psi \vdash \Sigma \), with \( \Sigma \equiv \Psi \)

  e.g., if \( \Sigma = P_1 \Leftrightarrow P_2 \)
  \( \Psi = \neg P_1 \Leftrightarrow \neg P_2 \)
  then \( \Sigma \equiv \Psi \) (why?)

  \( \Rightarrow \text{ truth table} \)

- \( \Sigma \) is valid if it is \( \vdash \) in every structure.

  e.g., \( P_1 \lor \neg P_1 \) is valid.
  while \( \downarrow \vdash P_1 \lor \neg P_1 \),
  or \( \not\exists \vdash P_1 \lor \neg P_1 \)

- \( \Sigma \) is satisfiable if there is a model of \( \Sigma \)

  \( \Sigma \) is satisfiable if there is a model of \( \Sigma \)
A deduction system

- above we define $\Sigma \vdash \phi$ ("$\phi$ follows semantically from $\Sigma$")

- now we will define $\Sigma \vdash \phi$ ("$\phi$ can be proved from $\Sigma$"
  "$\phi$ follows syntactically from $\Sigma$"

The deduction rule

$\Sigma \vdash \phi$ is a theory and $\phi, \Psi$ are sentences

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R1. $\phi, \psi \vdash \phi$

R2. If $\phi \vdash \chi \Psi$ for all $\chi$ and $\Psi$ implies $\chi$

R3. If $\phi \in \Sigma$ then $\Sigma \vdash \phi$
Examples of deduction (before formal definition)

\( \forall x \ (\neg \neg P_x \rightarrow P) \)

Why: \( \forall x \ (\neg \neg P_x \rightarrow P) \) by 1.

\[ \forall x \ (\neg \neg P_x \rightarrow P) \]

So: \( \forall x \ (\neg \neg P_x \rightarrow P) \) by 5 \( \Rightarrow \) \( \neg \neg P \)

\( \exists x \ (x \in \Sigma = \{ P_1, P_2 \}) \)

then \( \Sigma \vdash P_2 \)

Why:
1. \( \Sigma \vdash P_1 \Rightarrow P_2 \) by R3
2. \( P_1 \Rightarrow P_2 \Rightarrow P_1 \Rightarrow P_2 \) by 6 \( \Rightarrow \) \( \neg \neg P_2 \)
3. \( \Sigma \vdash P_1 \Rightarrow P_2 \) by R2, line 2
4. \( \Sigma \vdash P_1 \) by R3
5. \( \Sigma \vdash P_1 \Rightarrow P_2 \) by 5 \( \Rightarrow \) \( \neg \neg P_2 \)
6. \( \Sigma \vdash P_2 \) by R2, line 4, 6

Some of the deduction rules are redundant

ex: Suppose \( \Sigma \) is a theory and \( x \in \Sigma \).
Then:

1  \[(e3 + e) \rightarrow e\]  by 3  r-in
2  \[(e \rightarrow e) \rightarrow e\]  by 3  r-ar
3  \[e \rightarrow e\]  by \(R3\) and R2
4  \[e \rightarrow e\]  by 1  r-ar

This shows that the rule R3 is redundant: follows from other rules.

In any proof involving R3 could replace with a version of above (but would make for a longer proof)

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The above are examples of formal deductions (or justifications)

In general a justification is a sequence

\[\left( (\Pi_0, \mu_0), \ldots, (\Pi_n, \mu_n) \right)\]

where each \(\Pi_i\) is a theory

\(\mu_i\) is a subset

and \(\Pi_i + \mu_i\) follows from

\[\left( (\Pi_0, \mu_0), \ldots, (\Pi_n, \mu_n) \right)\]

and a deduction rule

1 single deduction every time!
Soundness

- Fix a theory $\Sigma$ and sentence $\varphi$

Q: what is the relation between

$\Sigma \vdash \varphi$ (purely syntactic)

and

$\Sigma \models \varphi$ (semantic)

- turns out these are equivalent, i.e.

$\Sigma \vdash \varphi$ iff $\Sigma \models \varphi$

- the forward direction $\Rightarrow$ is called Soundness of PL (if we can prove it from $\Sigma$, then it's true in every model of $\Sigma$)

surprisingly, $\Leftarrow$ is called completeness of PL (if it's true in every model, then we can actually prove it!)

Then (Soundness of PL) Fix a theory $\Sigma$ and sentence $\varphi$. If $\Sigma \vdash \varphi$, then $\Sigma \models \varphi$. 
DF: "obvious" (chem)
- Idea: in every deduction rule, can prove we can replace t with 1. theorem follows immediately (why?)
- e.g. consider "(\ell, \chi) \vdash \ell \land \chi"
  - Fix a structure A with A \models (\ell, \chi)
  then Truth_A(\ell) \land \text{Truth}_A(\chi) = 1
  hence Truth_A(\ell \land \chi) = 1
  hence A \models \ell \land \chi
  since A arbitrary, (\ell, \chi) \vdash \ell \land \chi.

- e.g. also consider "if \Sigma \cup \{\ell\} \vdash \chi then \Sigma \vdash \ell \Rightarrow \chi"

- Let's prove "if \Sigma \cup \{\ell\} \models \chi then \Sigma \vdash \ell \Rightarrow \chi"

\textit{DF. - assume} \Sigma \cup \{\ell\} \models \chi. \ \text{UFS} \ \Sigma \vdash \ell \Rightarrow \chi
- Fix A \models \Sigma
  - if \text{Truth}_A(\ell) = 0 \ then
    Truth_A(\ell \Rightarrow \chi) = 1 \ hence A \models \ell \Rightarrow \chi
  - if \text{Truth}_A(\ell) = 1
    then A \models \Sigma \cup \{\ell\} \ hence by assumption 
    A \models \chi, \ \therefore \ \text{Truth}_A(\chi) = 1.
- hence \text{Truth}_A(\ell \Rightarrow \chi) = 1
  hence \ A \models \ell \Rightarrow \chi.
In all cases \( A \vdash \varnothing \Rightarrow \gamma \).

Since \( A \vdash \varnothing \) was arbitrary,

\[ \Sigma \vdash \varnothing \Rightarrow \gamma, \text{ as desired.} \]

And likewise for the other deduction rules; we can turn any rule into a "\( \vdash \) version."

Then if we have some proof that

\[ \Sigma \vdash \varnothing, \text{ e.g.} \]

\[ \Gamma_0 \vdash \varnothing_0, \text{ because } (\cdots) \text{ deduction rule} \]

\[ \Gamma_1 \vdash \varnothing_1, \text{ because } (\cdots) \text{ deduction rule} \]

\[ \Gamma_m \vdash \varnothing_m, \text{ because } (\cdots) \text{ deduction rule} \]

\[ \Sigma \vdash \varnothing, \text{ because } (\cdots) \]

Can turn it into proof that \( \Sigma \vdash \varnothing \text{ by appropriate deduction rules with } \Gamma. \)

Therefore (i.e., deduction rules play many roles...)

So the rules of the faith (don't break it).
Every common to soundness is completeness:
if $\Sigma = \varepsilon$ then $\Sigma \vdash \varepsilon$

The idea is: our deduction rules are "sufficient" to turn any (informal) proof that $\Sigma = \varepsilon$ into a formal deduction that $\Sigma \vdash \varepsilon$.

Thus it is not how we will prove completeness (would involve quantifying over all informal arguments that $\Sigma = \varepsilon$).

We need some preliminary results:

Then (Finiteness of Deduction).
If $\Sigma \vdash \varepsilon$ then there is a finite subset $\Delta \subseteq \Sigma$ s.t. $\Delta \vdash \varepsilon$.

PF: "obvious" (short) because justifications are finite, and in any deductive step can refer to only finitely many sentences from previous steps.

really: we induct on the statement "if $\Sigma \vdash \varepsilon$ has a justification of length $n$, then there is a
Proof $\Delta \subseteq \Sigma$ s.t. $\Delta \vdash \varepsilon"
Observe, there are two types of deduction rules:

- **Absolute rules**:
  - Example: \( (x, y) \rightarrow \forall y \)

- **Conditional rules**:
  - If \( \exists u \forall e \rightarrow y \) then \( \exists f u e \rightarrow y \)

All absolute rules have finite sets on left of \( \vdash \).

No conditional rule can be used as first step in justification, except \( R_3 \).

Now we induce:

- If \( \vdash \Phi \) is given by deduction class \( E \), then \( E \) is finite.

\( \text{(BC)} \) Suppose \( \exists f u e \rightarrow y \) has a justification of length \( l \), then \( \exists f u e \rightarrow y \) as given by absolute rule \( \Phi \) so \( \exists f u e \rightarrow y \) is finite. (Or \( R_3 \), which can be dispensed with.)

\( \text{(IH)} \) Assume in any justification

\[
\Pi_0 \vdash \chi_0
\]

\[
\Pi_1 \vdash \chi_1
\]

of length \( n \) there exist finite \( N \in \mathcal{C}_k \) such that \( \Pi_0 \vdash \chi \) for every \( k \).
Now suppose
\[ \Gamma_0 + \alpha. \]

That len-
\[ \Sigma + \epsilon \]
\[ u \] is a justification of length \( n+1 \)

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- last step is justified by single deductor rule.
- If this is an absolute rule, then \( \Sigma \) is finite, so done.
- So suppose last step given by a conditional rule: \( 7 \rightarrow \neg\alpha, \exists - \alpha, \forall \alpha \neg \forall \alpha \rightarrow \neg \forall \alpha \). 
  \[ R1, R2, R3 \]

- e.g., if \( 7 \rightarrow \neg\alpha \), then \( \forall \alpha = 7 \forall \alpha \text{ for some } \forall \alpha \text{ and two previous lines.} \]
- In the justification, we are
  \[ \Sigma \cup \{\forall \alpha\} + \alpha \] 
  \[ \Sigma \cup \{\forall \alpha\} + \alpha \]

- by induction, there is finite \( C \in \Sigma \)
- e.g. \( \Sigma \cup \{\forall \alpha\} + \alpha \)
  \[ \Delta \cup \{\forall \alpha\} + \alpha \]

- hence \( \Delta \vdash \forall \alpha \) by \( 7 \rightarrow \neg \alpha \), i.e. \( \forall \alpha \)
- but \( \Delta \in \Sigma \) is finite, as desired.
Similar as works for other conditional deduction rules.

We'll need finitism of deduction to prove completeness of PL.

Some more terminology:

If \( \Sigma + \phi \) we say \( \phi \) is a theorem of \( \Sigma \).

\( \Sigma \) is consistent if \( \Sigma + \bot \)

is not a theorem of \( \Sigma \).

(\( \Sigma \) is inconsistent if \( \Sigma + \top \))

\( \Sigma \) is complete if for every sentence \( \phi \), either \( \Sigma + \phi \) or \( \Sigma + \neg \phi \).

Warning: "Complete" here has entirely different meaning than "completeness of PL."

"\( \Sigma \) is complete" means \( \Sigma + \phi \) or \( \Sigma + \neg \phi \) for every \( \phi \).

"Propositional Logic is complete" means for every theory \( \Sigma \) and sentence \( \phi \), if \( \Sigma \models \phi \) then \( \Sigma + \phi \)."
Theorem TFAE:

1. \( \Sigma + \bot \) (\( \Sigma \) is inconsistent)
2. For every \( \varphi \), \( \Sigma + \varphi \)
3. For some \( \varphi \), \( \Sigma + \varphi \) and \( \Sigma + \neg \varphi \)

PF: 1 \( \Rightarrow \) 2 - Suppose \( \Sigma + \bot \) and fix a sentence \( \varphi \)
- Then \( [\bot] + \varphi \) by 1 - out
- Hence \( \Sigma + \varphi \) by R2.

2 \( \Rightarrow \) 3 - Obvious: Fix any sentence \( \varphi \). Then by 2, \( \Sigma + \varphi \) and \( \Sigma + \neg \varphi \)

3 \( \Rightarrow \) 1 - Suppose there is \( \varphi \) s.t.
\( \Sigma + \varphi \) and \( \Sigma + \neg \varphi \)
- Then \( \Sigma + [\bot] + \varphi \) by R1
- And \( \Sigma + [\bot] + \neg \varphi \) by R1
- Hence \( \Sigma + [\bot] + \bot \) by 1 - out
- Hence \( \Sigma + \bot \) by R2

I don't get too caught up by particular deductive rules needed to prove this theorem, which are both outward
The point is: we just need some deductive system whose rules allow us to prove this in our system works, but isn't unique in this regard.

Theorem tells us: inconsistent theories are uninteresting (they prove everything).

In particular, inconsistent theories have no models (i.e. are unsatisfiable).
- If $\Sigma$ is inconsistent then $\Sigma \vdash P_0$ and $\Sigma \vdash \neg P_0$.
- Hence by soundness $\Sigma \vdash P_0$ and $\Sigma \vdash \neg P_0$.
- i.e. every model $A \models \Sigma$.
- must also have $A \models P_0$, $A \models \neg P_0$.
- Of course, there are no such structures $A$.
- Hence $\Sigma$ is unsatisfiable.

Another way to phrase this: if $\Sigma$ has a model, then $\Sigma$ is consistent.
What about consistent theories?
Next, we'll show every consistent $\Sigma$ has a model.
We need some preliminary first.

Lemma: Suppose $\Sigma$ is consistent and $\psi$ is a sentence.
Then either $\Sigma \vdash \psi$ or $\Sigma \not\vdash \psi$ is consistent.

Proof: Suppose not.
Then both $\Sigma \cup \{\psi\}$ and $\Sigma \cup \{\psi^c\}$ are inconsistent.

Hence by theorem
$\Sigma \cup \{\psi\} \vdash \psi$ and $\Sigma \cup \{\psi^c\} \vdash \psi$.
$\Sigma \vdash \psi$ and $\Sigma \vdash \psi^c$.
$\Sigma \vdash \psi \rightarrow \psi^c$.
$\Sigma \vdash \bot$.
$\Sigma$ is inconsistent, by theorem.

Contradiction, since we assumed $\Sigma$ was consistent.
Complete

**Completeness Lemma:** If \( \Sigma \) is consistent, then there is a consistent complete theory \( \Sigma' \) with \( \Sigma \subseteq \Sigma' \).

- Since \( \Sigma \) is consistent,
- we'll actually build \( \Sigma' \) so that for every \( \phi \), either \( \phi \in \Sigma' \) or \( \neg \phi \in \Sigma' \).
- Such a \( \Sigma' \) is clearly complete by R3.

\( \blacksquare \)

By a previous result, the set of sentences is countable, so we may enumerate them:
\[ \Sigma_0, \Sigma_1, \ldots, \]

Inductively define a sequence of theories \( \Sigma_n \):

- \( \Sigma_0 = \Sigma \)
- \( \Sigma_{n+1} = \Sigma_n \cup \{ \phi \} \) if \( \Sigma_n \) is consistent
- \( \Sigma_{n+1} = \Sigma_n \cup \{ \neg \phi \} \) if \( \Sigma_n \cup \{ \phi \} \) is inconsistent.
\[ \Sigma_0 = \Sigma \text{ is consistent by hypothesis} \]
and if \( \Sigma_n \) is consistent then either \( \Sigma_{n+1}(\Sigma_n) \) or \( \Sigma_{n+1}(\Sigma_n) \) is consistent by the model extension lemma.

- hence \( \Sigma_n \) is consistent for all \( n \). We have \( \Sigma_0 \subseteq \Sigma_1 \cdots \).

- let \( \Sigma' = \bigcup \Sigma_n \)

- Clearly \( \Sigma' \) is complete since \( \Sigma \subseteq \Sigma' \) or \( \Sigma \subseteq \Sigma' \) for every \( \Sigma \).

- but \( \Sigma' \) is also consistent.

Why? If not, then \( \Sigma + \top \) by induction of deductions.
There would be a finite subtheory \( \Delta \subseteq \Sigma \) s.t. \( \Delta + \top \).
- but if \( \Delta \) is finite then for some \( n, \Delta \subseteq \Sigma_n. \)
- hence \( \Sigma_n + \top \) by RI.
- contradiction.

hence \( \Sigma' \) is consistent and complete.
and \( \Sigma \subseteq \Sigma' \).
Consider $\Sigma = \{ P_0, P_1 \implies P_2, P_5 \}$
- $\Sigma$ is consistent (why?)
- Can extend $\Sigma$

Eventually get a complete consistent extension of $\Sigma$.

Recall: Soundness says $\Sigma$:
- if $\Sigma$ has a model
  then $\Sigma$ consistent

Here is the converse.
Theorem: If \( \Sigma \) is consistent, then \( \Sigma \) has a model.

**PF:** Suppose \( \Sigma \) is consistent.
- There is \( \Sigma' \supset \Sigma \) st.
  for every \( \psi \in \Sigma' \), either \( \psi \in \Sigma \) or \( \neg \psi \in \Sigma' \)
  and \( \Sigma' \) is consistent.

Claim 1: For every \( \psi \), exactly one of \( \psi \in \Sigma \) and \( \neg \psi \in \Sigma' \) holds.

**PF:** \( \neg \psi \) is inconsistent (why?)

Claim 2: For every \( \psi \), \( \Sigma' \vdash \psi \) if \( \psi \in \Sigma \).

**PF:** \( \leftarrow \)-direction given by R3
- Assume \( \Sigma' \vdash \psi \). Either \( \psi \in \Sigma \) or \( \neg \psi \in \Sigma' \).
  If \( \neg \psi \in \Sigma' \) then \( \Sigma' \vdash \neg \psi \)
  and \( \Sigma' \) is inconsistent, a contradiction.
  Hence \( \psi \in \Sigma \).

Define a structure \( A: \{P_i\} \rightarrow \{0, 1\} \) by
- \( A(P_i) = 1 \) if \( P_i \in \Sigma' \)
- \( A(P_i) = 0 \) if \( P_i \notin \Sigma' \).
Claim 3: Truth_A(ce) = 1 iff c ∈ E'

Proof: by induction on construction of E.

(BC) \( \text{Truth}_A(p_i) = \text{A}(p_i) = 1 \) iff \( p_i \in E' \)

- \( \text{Truth}_A(\mathbf{T}) = 1 \)
  and \( T \in E' \) since \( E' \models T \) (by def'n of A)

- \( \text{Truth}_A(\mathbf{⊥}) = 0 \)
  and \( \bot \not\in E' \) since \( E' \) is consistent (using R3 here)

(IH) Sps \( c, \chi \) are sentences and \( \text{Truth}_A(c_i) = 1 \) iff \( c_i \in E' \)
- \( \text{Truth}_A(\chi) = 1 \) iff \( \chi \in E' \)

Then \( \text{Truth}_A(\neg c) = 1 \) iff \( \text{Truth}_A(c) = 0 \) iff \( c \not\in E' \) (by IH)
- \( \text{Truth}_A(c \chi) = 1 \) iff \( c \chi \in E' \) (by Claim 1)

Now need to check \( \text{Truth}_A(c \chi) = 1 \)
- \( \text{if} \ c \chi \in E' \), then \( \chi \leftarrow t \Rightarrow \text{or} \leq \)
We check for $\land$:

\[
\text{Truth}_A(\epsilon_1 x) = 1 \iff \\
\text{Truth}_A(\epsilon) = 1 \text{ and } \text{Truth}_A(x) = 1 \iff \\
\epsilon \in \emptyset \text{ and } x \in \emptyset \quad (\text{by I4}) \iff \\
[\epsilon, x] \subseteq \emptyset
\]

Claim: $\epsilon \in \emptyset \iff x \in \emptyset$

**Proof:**

$(\Rightarrow)$ Assume $\{\epsilon, x\} \subseteq \emptyset$

Then $\emptyset + \epsilon x$ by $\land$-out.

$(\Leftarrow)$ Assume $\emptyset + \epsilon x$

Then $\emptyset + x \in \emptyset$ by $\land$-in, R1

So we continue:

$\iff \emptyset + \epsilon x$

$\iff \epsilon x \in \emptyset$ by Claim 2.

Similarly for $\lor$, $\Rightarrow$, $\Rightarrow$.

Hence by induction $\text{Truth}_A(1) = 1$ iff $\epsilon \in \emptyset$.

It follows that $A \models E$

hence $A \models \emptyset$. 

Proof: not crazy: given consistent \( \Sigma \), extend to consistent \( \Sigma' \) containing \( P_i \) or \( \neg P_i \) for every \( i \).
Define \( A(P_i) = 1 \) iff \( P_i \in \Sigma' \).
Then \( A \models \Sigma' \) (as long as \( \Sigma \models \Sigma' \)).

But theorem is powerful:

Theorem (Completeness of PL).

Suppose \( \Sigma \) is a theory and \( \phi \) a sentence. Then if \( \Sigma \models \phi \), then \( \Sigma \vdash \phi \).

**PF:** By contraposition.

- Suppose statement \( \neg \phi \) is trivial if \( \Sigma \) is inconsistent, so assume \( \Sigma \) is consistent and \( \neg \phi \).

Claim \( \Sigma + \neg \phi \models \phi \)

Why: if \( \Sigma + \neg \phi \) is inconsistent, then \( \Sigma + \neg \phi \vdash \phi \).

\( \Sigma + \neg \phi \vdash \neg \phi \) for some \( \psi \).

\( \Sigma + \neg \phi \vdash \psi \) by 7-in

\( \Sigma + \neg \phi \vdash \neg \psi \) by 7-out

\( \Sigma + \neg \phi \vdash \neg \neg \phi \) by \text{contradiction}
- Here we claim as claimed.
- Here there is a model \( A = \{ a, b, c \} \).
- Here \( A = \emptyset \) and \( A \neq \emptyset \).
- But then \( \emptyset \neq \emptyset \), there is proved.\( \checkmark \)

**Let** \( \Sigma = \{ P_0, P_0 \rightarrow P_1, P_1 \rightarrow P_2, P_2 \rightarrow P_3 \} \)

Then \( \emptyset \rightarrow P_3 \)

**Pf**: We prove \( \emptyset \rightarrow P_3 \).
- If \( A = \emptyset \), then \( \text{Truth}_A(P_0) = 1 \) since \( \text{Truth}_A(P_0) = 1 \) due to \( \text{Truth}_A(P_2) = 1 \) and \( \text{Truth}_A(P_1) = 1 \) due to \( \text{Truth}_A(P_3) = 1 \).

So since \( A \) arbitrary, \( \emptyset \rightarrow P_3 \).

Point: Our deduction system powerful enough to derive these lemmas we found deducible.
Here is a useful corollary of the completeness theorem.

**Compactness Theorem:** Suppose $\Sigma$ is a theory and every finite subset $\Delta \subseteq \Sigma$ has a model. Then $\Sigma$ has a model.

**Proof:** Toward a contradiction, suppose $\Sigma$ has no model.

Then by Completeness Theorem, $\Sigma$ is inconsistent.

But why is $\Sigma$ inconsistent?

For $\Sigma$ has a model.

Here $\Sigma \vdash \bot$.

By finiteness of deductivity, there is a finite subset $\Delta \subseteq \Sigma$.

But $\Delta \vdash \bot$.

But then $\Delta$ has no model.

A contradiction.

Here $\Sigma$ has a model.