Then says: \( N \) is as small as possible. For an infinite set: even sets that appear smaller (e.g. \( \mathbb{E} \), \( 0 \), or other infinite subsets of \( N \)) are actually not.

OTOH: many sets which appear larger than \( N \) (e.g. \( \mathbb{I} \), \( \mathbb{Q} \)) are actually the same size.

**Def'n:** A set \( X \) is countable if \( \exists x \in X \) such that \( X \subseteq \mathbb{Z} \) is countable.

**Ex's:** \( \mathbb{Z} \) is countable.

**PF:** we already showed

\[ f: \mathbb{Z} \rightarrow N \quad f(n) = \begin{cases} 2n & n > 0 \\ 2(n-1) + 1 & n \leq 0 \end{cases} \]

is a bijection.

\( \mathbb{N} \times \mathbb{N} \) is countable

**PF:** need to construct a bijection

\[ f: \mathbb{N} \rightarrow \mathbb{N} \times \mathbb{N} \]

possible to do this explicitly (i.e., define \( f \) w/ a formula), but we just draw a picture.
to construct \( f: N \to N \times N \) we "count \( N \times N \) along its diagonals" 

\[
\begin{array}{cccc}
(1,1) & (2,2) & \cdots & N \times N \\
(1,1) & (2,1) & \cdots & \end{array}
\]

\[
\begin{array}{cccc}
\circ & \circ & \cdots & \\
\circ & \circ & \cdots & \\
\end{array}
\]

\[
\begin{array}{cccc}
1 & 3 & \cdots & 10 \\
\end{array}
\]

resulting \( f: N \to N \times N \) is injective + surjective

**Theorem (Cantor-Schröder-Bernstein)**

For any sets \( A, B \)

if \( A \subseteq B \) and \( B \subseteq A \) then \( A = B \).

**PF:** interesting but tricky — we'll skip and take theorem for granted.

The point: sometimes it's hard to show \( A \neq B \) directly (i.e., build a bijection) but easier to show \( A \subseteq B \) and \( B \subseteq A \) then says: they are the same.
Note: CSB says \( ASB \wedge BSA \Rightarrow A \cap B \) (16)

Since we know: \( ASB \iff BZA \), CSB also gives that: \( BZA \wedge A \cap Z \Rightarrow A \cap B \). I.e., \( \subseteq \) and \( \supseteq \) are "antisymmetric up to \( n \)."

Next goal: prove that \( N \nsubseteq Q \! \). 

First need: Theorem: if \( A, B \) are countable sets then \( A \times B \) is countable.

PF: Supp \( A, B \) are countable, i.e., we have bijections \( F: N \rightarrow A \) 
\[ g: N \rightarrow B. \]

We know: \( N \nsubseteq N \times N \)

So if we can show: \( N \times N \nsubseteq A \times B \) we'll be done (by transitivity of \( \subseteq \)).

Consider \( F: N \times N \rightarrow A \times B \)
defined by \( F(n, m) = (f(n), g(m)) \)
Claim F is a bijection.

PF (surj) - Fix \((a,b) \in A \times B\).
- Since \(F, g\) both surj., \(\exists y \in N\) s.t.
  \[ F(n) = a \quad F(m) = b \]
- hence \(F(n,m) = (a,b)\)

PF (inj.) - if \(F(n,m) = F(n',m')\)
  then \((F(n), g(m)) = (F(n'), g(m'))\)
- i.e. \(F(n) = F(n')\) and \(g(m) = g(m')\)
- since \(F, g\) both surj. this implies \(n = n'\) and \(m = m'\)
- i.e. \((n,m) = (n',m')\)

hence \(N \times N \sim A \times B\)

hence \(N \sim A \times B\), as desired.

Theorem: \(\mathbb{Q} \cup \mathbb{R} = \mathbb{R}\), i.e. \(\mathbb{N} \sim \mathbb{Q}\).

PF: we'll prove: \(\mathbb{N} \preceq \mathbb{Q} \preceq \mathbb{Z} \times \mathbb{N} \preceq \mathbb{N}\)

By transitivity this gives \(\mathbb{N} \preceq \mathbb{R} \preceq \mathbb{N}\)
which by CSSB gives \(\mathbb{N} \sim \mathbb{Q}\).
(1) holds since $\mathbb{Q}$ is infinite, by our preview theorem.

(2) holds by the preceding theorem: we knew $\mathbb{Z} \times \mathbb{N}$ and $\mathbb{N} \times \mathbb{N}$, hence $\mathbb{Z} \times \mathbb{N} \times \mathbb{N}$.

So it remains to prove $\mathbb{Q}$ to prove $\mathbb{Z} \times \mathbb{N}$ version.

Claim: $\mathbb{Z} \times \mathbb{N} \supseteq \mathbb{Q}$

Proof: $F : \mathbb{Z} \times \mathbb{N} \rightarrow \mathbb{Q}$ defined by $F(mn) = \frac{m}{n}$ is a surjection

(why: given $q \in \mathbb{Q}$, if $q = \frac{m}{n}$ then $F(mn) = q$!)

Note: $F$ is not injective: e.g. $F(12) = F(24) = ...$

Who cares! We've still shown $\mathbb{Z} \times \mathbb{N} \supseteq \mathbb{Q}$.

Hence $\mathbb{Q} \subseteq \mathbb{Z} \times \mathbb{N}$.

Combined with above, this gives $\mathbb{Q} = \mathbb{N}$.

Discussion: Among infinite sets $x$, $\mathbb{N}$ is "small" in the sense that $\mathbb{N} \times \mathbb{N}$ is always larger than $\mathbb{N}$ for many sets $x$ which appear larger than $\mathbb{N}$ (e.g. $\mathbb{Z}, \mathbb{Q}$). We actually have $\mathbb{Z} \times \mathbb{N}$.

Q: Are there infinite sets $x$ for which $\mathbb{Q} \times \mathbb{N}$?
Yes!

Theorem (Cantor): $\mathbb{N} < \mathcal{P} \mathbb{N}$
That is: $\mathbb{N} \leq \mathcal{P} \mathbb{N}$ but $\mathbb{N} \nleq \mathcal{P} \mathbb{N}$

Proof: we know $\mathbb{N} \leq \mathcal{P} \mathbb{N}$ since $\mathcal{P} \mathbb{N}$ is infinite.
So need to show $\mathbb{N} \nleq \mathcal{P} \mathbb{N}$

Claim: Suppose $F: \mathbb{N} \to \mathcal{P} \mathbb{N}$ is a fixed function (any function).
Then $F$ is not surjective.

Proof: a magic trick.

Let $T = \{ n \in \mathbb{N} | n \notin F(n) \}$

To illustrate definition, e.g. if

- $F(1) = \{1, 7, 103\}$
- $F(2) = \{1, 3, 5, 7, \ldots \}$
- $F(3) = \emptyset$
- $F(4) = \{2, 4, 6, 8, \ldots \}$

Then
- $1 \notin T$ since $1 \in F(1)$
- $2 \in T$ since $2 \notin F(2)$
- $3 \notin T$ since $3 \notin F(3)$
- $4 \notin T$ since $4 \notin F(4)$ etc.

So $T = \{2, 3, \ldots \}$ in this case.
Then: \((\forall n \in \mathbb{N}) \ f(n) \neq T\)

**Proof:** Fix \(n \in \mathbb{N}\).

(i) If \(n \in T\), then \(n \notin f(n)\), by def'n of \(T\). Hence \(f(n) \neq T\), since \(n \notin f(n)\).

(ii) If \(n \notin T\), then \(n \in f(n)\), by def'n of \(T\).

Hence \(f(n) \neq T\) in this case too: \(n \notin f(n)\), \(n \notin T\).

Hence in all cases, \(f(n) \neq T\). Since \(n\) was arbitrary, we have \(f(n) \neq T\) for every \(n \in \mathbb{N}\).

But now the claim follows: \(T \neq \text{Im} f\), hence \(f\) is not surjective.

Since \(f\) was arbitrary, there is no surjection \(f: \mathbb{N} \to \mathcal{P}(\mathbb{N})\) (hence no bijection).

(\(\mathbb{N} \neq \mathcal{P}(\mathbb{N})\)).

The same proof works in general.

**Theorem:** For any set \(A\), there is no surjection \(f: A \to \mathcal{P}(A)\).

**Proof:** Fix \(f: A \to \mathcal{P}(A)\) and let \(T = \{x \in A \mid x \notin f(x)\}\).

Then \(\forall x \in A\), \(f(x) \neq T\) (by some arg).
However, it's always the case that $A \subseteq P(A)$ (since a 3 sets defines an injection). So above theorem shows that $A \subseteq P(A)$ for every $A$.

It follows that there are infinitely many levels of infinity!

$N < P(N) < P(P(N)) < \ldots$

**Defn** If $x$ is infinite and $N \times x$ we say $x$ is uncountable.

So, by above: $P(N)$ is unctable. Other examples?

**Sets of Functions**: Consider the set $F$ of functions $F: N \rightarrow \{0, 1\}$.

i.e. $F = \{ f \in \mathbb{N} \times \{0, 1\} \mid f \text{ is a function} \}$

We can think of a given $F \in F$ as an infinite 01-sequence.

e.g. if $f(1) = 0$ $f(5) = 1$ $f(2) = 0$ $f(6) = 0$ $f(3) = 1$ $f(4) = 0$ Can picture $F$ like this:

$F = " 001010 \ldots "$
Conversely, could write:
\[ g^* \approx 101010 \ldots \]
to mean that \( g \) is the function
\[
    \begin{align*}
    g(1) &= 1 \\
    g(2) &= 0 \\
    g(3) &= 1 \\
    g(4) &= 0 \\
    &\quad \text{etc.}
\end{align*}
\]

**Theorem.** \( F \) is uncomputable.

**Proof.** Diagonalize!

**Claim:** If \( H: \mathbb{N} \rightarrow F \) is a function then \( H \) is not surjective.

**Proof:** consider the function \( F: F \)

*defined as follows:
\[
    F(n) = \begin{cases} 
    1 & \text{if } H(n)(n) = 0 \\
    0 & \text{if } H(n)(n) = 1 
    \end{cases}
\]

then, by the very defn of \( F \) we have that \((\forall n \in \mathbb{N}) \ F(n) \neq H(n)(n)\)

* hence: \((\forall n \in \mathbb{N}) \quad F \neq H(n)\)

\( F \) and \( H(n) \) differ in the \( n \)th output.
The claim follows.

to illustrate:

E.g. if we have:

\[ H(1) = 01010 \ldots \]
\[ H(2) = 001101 \ldots \]
\[ H(3) = 111111 \ldots \]
\[ H(4) = 000011 \ldots \]

In this case we'd have

\[ F = 1100 \ldots \]

Observe: \( F \neq H(n) \) for any \( n! \) Why: \( F(n) \neq H(H(n)) \).