Again we have strict containment in this case. 

\[ \text{Imp(PreImp}(y)) = 113 \land [2,1] = y. \]

\underline{Junctions:} \quad \text{We have } A = \{1,2,3\}

\[ B = \{\ast, 0\} \]

\[ C = \{1, 2\} \]

\[ D = \{\ast, 0, \Delta\} \]

\underline{Define}

\[ g: A \rightarrow B \]

\[ h: C \rightarrow D \]

\[ j: A \rightarrow D \]

\underline{by:}

\[ g = \{(1, \ast), (2, 0), (3, \ast)\} \]

\[ h = \{(1, \ast), (2, 0)\} \]

\[ j = \{(1, \ast), (2, 0), (3, \Delta)\} \]

\[ A \quad B \]

\[ 1 \quad 2 \quad 3 \]

\[ C \quad D \]

\[ 1 \quad 2 \quad 3 \]

\[ A \quad D \]
Def: A function $f: A \rightarrow B$ is **surjective** (or **onto**, or **a surjection**) iff $\text{Im} f = B$.

i.e., iff

$$(\forall b \in B)(\exists a \in A)(f(a) = b)$$

- ex'g and j above are surjective
- h is not, since $A \notin \text{Im}_h$

**Proving surjectivity:**

**Ex:** 1) Define $f: \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Z}$ by $f(mn) = mn$.

**Claim:** $f$ is surjective

**PF:** WTS $(\forall x \in \mathbb{Z})(\exists (a,b) \in \mathbb{Z} \times \mathbb{Z})(f(a,b) = x)$

- So fix $x \in \mathbb{Z}$
- observe $f(c,x) = c + x = x$
- hence $f(c,x) \in \mathbb{Z} \times \mathbb{Z}$ s.t. $f(c,x) = x$ namely $(c,x) = (c,x)$
- Since $x$ was arbitrary, claim is proved!
2. Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = 2x + 1$.

   **Claim:** $f$ is surjective

   **PF:** Fix $y \in \mathbb{R}$.
   - Let $x = \frac{y-1}{2}$
   - Then $f(x) = f\left(\frac{y-1}{2}\right) = 2\left(\frac{y-1}{2}\right) + 1 = y$
   - Since $y$ was arbitrary, claim is proved.

3. Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^2$.

   **Claim:** $f$ is not surjective

   **PF:** Let $(\forall y \in \mathbb{R}) (\exists x \in \mathbb{R}) (f(x) = y)$
   - i.e. $(\forall y \in \mathbb{R}) (\forall x \in \mathbb{R}) (f(x) \neq y)$
   - Let $y = -1$: Then fix $x \in \mathbb{R}$.
   - Observe $f(x) = x^2 \geq 0 > -1$
   - $\Rightarrow f(x) \neq -1$.

   Hence, since $x$ was arbitrary,
   $(\forall x \in \mathbb{R}) (f(x) \neq -1)$

   Hence, $-1 \notin \operatorname{Im} f \Rightarrow f$ is not surjective.
Def: a function \( F : A \rightarrow B \) is **injective** (or one-to-one, or 1-1, or an injection), if

\[(\forall x, y \in A) \quad (f(x) = f(y) \Rightarrow x = y)\]

Equivalently:

\[(\forall x, y \in A) \quad (x \neq y \Rightarrow f(x) \neq f(y))\]

"distinct inputs map to distinct outputs"

**Ex's** - \( g \) above is **not** injective since \( 1 \neq 3 \) but \( g(1) = g(3) = * \)

- \( h, j \) are injective.

**Proving Injectivity**

Two approaches: Fix \( x, y \in A \) and either:
- Assume \( f(x) = f(y) \), prove \( x = y \)
- Assume \( x \neq y \), prove \( f(x) \neq f(y) \)

**Ex's**

Consider again \( F : \mathbb{R} \rightarrow \mathbb{R} \)

defined by \( f(x) = 2x + 1 \).

**Claim:** \( F \) is injective.

**Pf:** - fix \( x, y \in \mathbb{R} \)
- assume \( f(x) = f(y) \)

\(- \)
-1.e. 2x + 1 = 2y + 1
\Rightarrow 2x = 2y
\Rightarrow x = y \checkmark \text{ since } x, y \text{ arbitrary, claim is proved.}

2. Define \( F : \mathbb{N} \to \mathbb{N} \) by \( F(n) = n^2 \)

Claim: \( F \) is injective.

Proof: Fix \( n, m \in \mathbb{N} \) and suppose \( n \neq m \)

WTS: \( F(n) \neq F(m) \)

Two cases:
1. \( m < n \)
2. \( n < m \)

If 1: Since \( n, m \) both positive, we can square both sides of inequality to get:
\[ m^2 < n^2 \]

So in particular \( F(m) \neq F(n) \)

If 2: Similar.

Since \( n, m \) were arbitrary, claim is proved.
Define: \( f: \mathbb{Z} \to \mathbb{Z} \) by \( f(n) = n^2 \)

Claim: \( f \) is not injective

PF: \( f(-2) = f(2) = 4 \)
\[ \text{but } -2 \neq 2 \]

**Def'n:** A function \( f: A \to B \) is **bijective** (or a **bijection**) iff \( f \) is both injective and surjective.

**Ex's:**
- \( g \) above is not bijective
  (surjective, but not injective)

- \( h(n) = n \)
  (injective, but not surjective)

- \( j \) is bijective

**Proving bijectivity:**

**Ex's:**
- Consider again \( f: \mathbb{R} \to \mathbb{R} \)
  defined by \( f(x) = 2x + 1 \).

Claim: \( f \) is bijective

Pf: we've already shown \( f \) is both
injunctive and surjective.
A spacy one: define \( f : \mathbb{Z} \rightarrow \mathbb{N} \) by:

\[
f(n) = \begin{cases} 
2n & \text{if } n > 0 \\
2(-n) + 1 & \text{if } n \leq 0
\end{cases}
\]

Claim: \( f \) is bijective.

**Proof (surjectivity):**
- Fix \( n \in \mathbb{N} \)
- If \( n \) is even, then \( n = 2k \) for some \( k \in \mathbb{N} \) (hence \( k > 0 \))
- Hence \( f(k) = 2k = n \)
- If \( n \) is odd, then \( n = 2k + 1 \) for some \( k \in \mathbb{N} \cup \{0\} \)
  (hence \( k \geq 0 \), hence \( -k \leq 0 \))
  - Hence \( f(-k) = 2k + 1 = n \)
In either case: \((\exists x \in \mathbb{Z}) f(x) = n\)

- hence \(F\) is surjective \(\checkmark\)

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**Injectivity**
- Fix \(n, m \in \mathbb{Z}\) and assume \(n \neq m\).
- We will \(F(n) \neq F(m)\)
- We will assume \(n < m\), since case when \(m < n\) is similar.

**Case 1:** \(0 < n < m\).
- then \(F(n) = 2n < 2m = F(m)\)
- hence \(F(n) \neq F(m)\)

**Case 2:** \(n < m \leq 0\).
- then \(F(n) = 2(-n) + 1\)
  \(F(m) = 2(-m) + 1\)
- observe: since \(n < m\)
  \[\begin{align*}
  -n &> -m \\
  2(-n) + 1 &> 2(-m) + 1
  \end{align*}\]
- i.e. \(F(n) > F(m)\)
- hence \(F(n) \neq F(m)\).

**Case 3:** \(n \leq 0 < m\)
- then \(F(n) = 2(-n) + 1\) is odd
  \(F(m) = 2m\) is even.
- hence $f(n) \neq f(m)$ in this case as well.
- hence in all cases $f(n) \neq f(m)$
- since $n,m$ were arbitrary, we've proved $F$ is injective
- hence $F$ is bijective

Compositions:

Def'n: Sup $F: A \rightarrow B$ and $g: B \rightarrow C$ are functions. The composition of $F$ and $g$, denoted $g \circ f$, is defined by:

$$(x \in A) \quad g \circ f(x) = g(F(x))$$

Observe: So defined, we see that $g \circ f$ is a function from $A$ to $C$. 