Then: 
\[ F(1) = 12 = g(1) \]
\[ F(2) = 30 = g(2) \]
\[ F(3) = 60 = g(3) \] hence \( F = g \! \)

1.e. \( F = \{(1,12),(2,30),(3,60)\} = g \)

**What is the magic trick?**

observe: \( F - g = x^3 - 6x^2 + 11x - 6 \)
\[ = (x-1)(x-2)(x-3) \]
\[ = 0 \text{ for } x \in \{1,2,3\} \]

**Images**

**Def'n Sps:** \( F: A \to B \) is a function and \( X \subseteq A \). 

The **image of** \( X \) under \( F \), denoted \( \text{Im}_f(X) \), is defined as:

\[ \text{Im}_f(X) = \{ b \in B \mid (\exists a \in X) \quad f(a) = b \} \]

Informally we can write:

\[ \text{Im}_f(X) = \{ f(a) \mid a \in X \} \]

The point is: \( a \in X \iff f(a) \in \text{Im}_f(X) \)

**Note:** when \( X = A \) itself, we say \( \text{Im}_f(A) \) is the **image of** \( f \) and sometimes just write \( \text{Im}_f \).
So defn says: \(-\text{Imp}(x)\) is the "set of outputs of elements in \(x\）" 
\(-\text{Imp} = \text{Imp}(A)\) is the "set of all outputs of \(F\)."

**Pic:**

\[
A \xrightarrow{f} B \\
\text{Imp}(A) = \text{Imp} \text{Imp}(x) \\
\text{Imp}(A) = \text{Imp} \text{Imp}(x)
\]

**Exs:** 0) \(A = \{1, 2, 3\}, \quad B = \{\ast, 0, \Delta\}\)

define \(F: A \rightarrow B\) by \(F = \{(1, \ast), (2, 0), (3, \ast)\}\)

Then: \(-\text{Imp}(\{1, 3\}) = \{F(1), F(3)\}\)

\[= \{\ast, \ast\}\]

\[= \{\ast\}\]

and: \(-\text{Imp} = \text{Imp}(\{1, 2, 3\}) = \{F(1), F(2), F(3)\}\)

\[= \{\ast, 0, \ast\}\]

\[= \{\ast, 0\}\].
Define $F: \mathbb{R} \to \mathbb{R}$ by $F(x) = x^2$.

Then: $\text{Imp}(\{\{1, 0, 1\}\}) = \{(-1, 0, 1)\}
= \{0, 1\}

- $\text{Imp} = \{x \in \mathbb{R} \mid x > 0\}$.

Functions add a layer of complexity to the basic set theory of $\cup, \cap, \ldots$ we studied before.

**Prop'n:** $\text{Sps } F: A \to B$ is a function and $S, T \subseteq A$.

Then $\text{Imp}(S \cup T) \subseteq \text{Imp}(S) \cup \text{Imp}(T)$

**PF:**

- Fix $y \in \text{Imp}(S \cup T)$
- Then $\exists x \in S \cup T$ s.t. $F(x) = y$.
- Hence $x \in S$ and $x \in T$
- Hence $F(x) \in \text{Imp}(S)$ and $F(x) \in \text{Imp}(T)$
- i.e. $y \in \text{Imp}(S)$ and $y \in \text{Imp}(T)$
- i.e. $y \in \text{Imp}(S) \cup \text{Imp}(T)$

Since $y$ was arbitrary the prop'n is proved.
Note: in general we don't have:
\[ \text{Imp}(S \cup T) = \text{Imp}(S) \cap \text{Imp}(T) \]

E.g. consider \( F(x) = x^2 \) on \( \mathbb{R} \).

Let \( S = [-1, 0] \quad T = [0, 1] \)

Then: \[ \text{Imp}(S) = \{ F(-1), F(0) \} = \{ 1, 0 \} \]
\[ \text{Imp}(T) = \{ F(0), F(1), F(2) \} = \{ 0, 1, 4 \} \]

\[ \Rightarrow \text{Imp}(S) \cap \text{Imp}(T) = \{ 0, 1 \} \]

(\text{counterexample}) \[ \text{Imp}(S \cup T) = \text{Imp}(T) \]
\[ = \{ F(0) \} = \{ 0 \} \]

So in this case: \[ \text{Imp}(S \cup T) \neq \text{Imp}(S) \cap \text{Imp}(T) \]

The essence of the issue: Functions can send multiple inputs to the same output.

Schematically:

\[ \text{In } \text{Imp}(S) \cap \text{Imp}(T) \]
\[ \text{but not } \text{Imp}(S \cup T) \]
**Pre-Image**

**Def'n:** Sups $F: A \to B$ is a function a $Y \subseteq B$. The preimage of $y$ under $F$, denoted $\text{PreImp}(y)$, is defined as:

$$\text{PreImp}(y) = \{ x \in A \mid F(x) \in Y \}$$

$=$ the set of inputs in $A$ whose outputs land in $Y$.

**Note:** Since $F(x) \in B$ for every $x \in A$, we don't separately define $\text{PreImp}(\emptyset)$ - this is always just $A$.

**Ex:** \( A = \{1, 2, 3\} \)

\( B = \{\ast, C, D\} \)

\( F = \{(1, \ast), (2, C), (3, \ast)\} \)

Then:

$$\text{PreImp}(\{\ast\}) = \{x \in A \mid F(x) \in \{\ast\}\}$$

$=$ \{ $x \in A \mid F(x) = \ast$ \}

$=$ \{1, 3\}
\[ \text{PreImp}(\{*,0\}) = \{ x \in A \mid f(x) \in \{*,0\} \} = \{1,2,3\} = A \]

\[ \text{PreImp}(\{0,3\}) = \{ x \in A \mid f(x) \in \{0,3\} \} = \{ x \in A \mid f(x) = 0 \} = \emptyset. \]

2. Consider \( f: \mathbb{R} \to \mathbb{R} \)

\[ f(x) = x^2 \]

Then:

\[ \text{PreImp}(\{0,1\}) = \{ x \in \mathbb{R} \mid f(x) \in \{0,1\} \} = \{ x \in \mathbb{R} \mid x^2 \in \{0,1\} \} = \{ \pm 1, 0, 1 \} \]

\[ \text{PreImp}(\{0,2\}) = \{ x \in \mathbb{R} \mid x^2 \in \{0,2\} \} = \{ x \in \mathbb{R} \mid 0 \leq x^2 \leq 2 \} = \{ x \in \mathbb{R} \mid x^2 \leq 2 \} = \{ x \in \mathbb{R} \mid -\sqrt{2} \leq x \leq \sqrt{2} \} = [-\sqrt{2}, \sqrt{2}] \]
\[ \text{PreImp}(C_{0, \infty}) = \{x \in \mathbb{R} \mid x^2 \in C_{0, \infty}\} \]
\[ = \mathbb{R} \]

Q: What happens if we take the preimage of the image of some \( x \in A \)?
   or the image of the preimage of some \( y \in B \)?

Prop\( \text{'} \): Sps \( f: A \rightarrow B \) is a function.

(i) Fix \( x \in A \).
   Then: \( \text{PreImp}(\text{Imp}(x)) = x \)

(ii) Fix \( y \in B \).
   Then: \( \text{Imp}(\text{PreImp}(y)) \leq y \).

PF: (i) Fix \( x \in X \).
   By defn: \( \text{PreImp}(\text{Imp}(x)) \)
   \[ = \{y \in A \mid f(y) \in \text{Imp}(x)\} \]
   but since \( x \in X \), we knew \( f(x) \in \text{Imp}(x) \)
   by defn of \( \text{Imp}(x) \)
   Hence \( x \in \text{PreImp}(\text{Imp}(x)) \)
   Since \( x \) was arbitrary, (i) is proved.
(iii) Now fix $y \in \text{Im}_f(\text{PreImage}(Y))$

by def'n of image, $\exists x \in \text{PreImage}(Y)$

S.L. $f(x) = y$.

But then, by def'n of preimage, $f(x) \in Y$, i.e., $y \in Y$.

Since $y$ was arbitrary, (iii) is proved.

**Picture:**

(iii) $\text{PreImage}(\text{Image}(x))$

Note: in general, neither containment can be reversed.

Ex: Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2$
Let \( X = [1, 3] \)

Then: \( \text{Imp}(X) = \text{Imp}( [1, 3] ) = \{ f(1) \} = \{ 1 \} \)

So then: \( \text{PreImp}(\text{Imp}(X)) = \text{PreImp}( [1, 3] ) = \{ x \in \mathbb{R} | f(x) \in [1, 3] \} = \{ x \in \mathbb{R} | x^2 \in [1, 3] \} = \{ x \in \mathbb{R} | x^2 = 1 \} = \{-1, 1\} \)

So we have strict containment in this case: \( X = [1, 3] \neq [-1, 1] = \text{PreImp}(\text{Imp}(X)) \)

Now let \( Y = [-2, 1] \)

Then: \( \text{PreImp}(Y) = \{ x \in \mathbb{R} | f(x) \in [-2, 1] \} = \{ x \in \mathbb{R} | x^2 \in [-2, 1] \} = \{-1, 1\} \)

So then: \( \text{Imp}(\text{PreImp}(Y)) = \text{Imp}( [-1, 1] ) = \{ (2, 4), f(1) \} = \{ [-1, 1] \} \neq [1, 1] \).
Again we have strict containment in this case.

\[ \text{Imp}(\text{PreImp}(Y)) = \{1\} \notin \{2,1\} = Y. \]

\begin{itemize}
  \item \text{SetOne}:
  \begin{align*}
    W & \vdash A = \{1, 2, 3\} \\
    B & = \{\ast, \heartsuit, \spadesuit\} \\
    C & = \{1, 2\} \\
    D & = \{\ast, \heartsuit, \Delta\}
  \end{align*}
\end{itemize}

define
\begin{align*}
  g : & A \rightarrow B \\
  h : & C \rightarrow D \\
  j : & A \rightarrow D
\end{align*}

by:
\begin{align*}
  g = \{(1, \ast), (2, \heartsuit), (3, \ast)\} \\
  h = \{(1, \ast), (2, \spadesuit)\} \\
  j = \{(1, \ast), (2, \heartsuit), (3, \Delta)\}
\end{align*}