(iii) Fix \( x, y, z \in \mathbb{R} \). Suppose \( (xy, xz) \in R \) and \( (y, z) \in R \). Then \( L_{xy} = L_y \) and \( L_{xz} = L_z \). Hence \( L_{xy} = L_{xz} \). Hence \( (x, z) \in B \times R \). 

(3) Define a relation \( \equiv_3 \) on \( \mathbb{Z} \) by \( (x, y) \in \equiv_3 \) if and only if \( 3 \mid (x - y) \). 

\[ \equiv_3 = \{(x, y) \in \mathbb{Z}^2 \mid 3 \mid (x - y) \} \]

We will typically write \( n \equiv_3 m \) instead of \( (n, m) \in \equiv_3 \). 

E.g. \( 2 \equiv_3 5 \) since \( 3 \mid (5 - 2) \)
\( 7 \equiv_3 -2 \) since \( 3 \mid (7 - (-2)) \)
\( 6 \not\equiv_3 7 \) since \( 3 \not\mid (7 - 6) \)

Claim \( \equiv_3 \) is an equivalence relation on \( \mathbb{Z} \).

PF: (i) Fix \( n \in \mathbb{Z} \). Observe \( 3 \mid (n - n) \), i.e. \( 3 \mid 0 \) since \( 0 = 3 \cdot 0 \). Hence \( n \equiv_3 0 \).

(ii) Fix \( n, m \in \mathbb{Z} \) and suppose \( n \equiv_3 m \). We prove \( m \equiv_3 n \).
PF: Since \( n \equiv_3 m \) we have \( 3 | m - n \)

\( \exists k \in \mathbb{Z} \) s.t. \( m - n = 3k \)

but then \( n - m = 3(-k) \)

hence \( 3 | n - m \)

hence \( \emptyset \) \( m \equiv_3 n \)

(iii) Fix \( n, m, l \in \mathbb{Z} \). Sup \( n \equiv_3 m \) and \( m \equiv_3 l \). We prove \( n \equiv_3 l \).

PF: we know \( \exists k_1, k_2 \in \mathbb{Z} \) s.t.

\[
\begin{align*}
m - n &= 3k_1 \\
l - m &= 3k_2
\end{align*}
\]

adding these equations gives:

\[
(m - n) + (l - m) = 3(k_1 + k_2)
\]

\( l - n = 3(k_1 + k_2) \)

\( 3 | l - n \)

Hence \( n \equiv_3 l \)

\( \equiv_3 \) is called congruence modulo 3.

It is more convenient to write

\( n \equiv m \) (mod 3) instead of \( n \equiv_3 m \)

(we'll use these notations interchangeably)
Another way to think about it:

\[ n \equiv m \pmod{3} \iff n, m \text{ have the same remainder when divided by } 3. \]

E.g. \[ 2 \equiv 5 \pmod{3} \]

Since \[ 2 = 3 \cdot 0 + 2 \text{ and } 5 = 3 \cdot 1 + 2 \text{ have same remainder} \]

\[ 7 \equiv 13 \pmod{3} \]

Since \[ 7 = 3 \cdot 2 + 1 \text{ and } 13 = 3 \cdot 4 + 1 \text{ have same remainder} \]

\[ 7 \equiv -2 \pmod{3} \]

Since \[ 7 = 3 \cdot 2 + 1 \text{ and } -2 = 3(-1) + 1 \]

but \[ 7 \not\equiv 11 \pmod{3} \]

Since \[ 7 = 3 \cdot 2 + 1 \text{ and } 11 = 3 \cdot 3 + 2 \text{ have different remainders} \]

Nothing special about 3. For any fixed \( k \in \N \), can define \( \equiv_k \) on \( \Z \) by:

\[ n \equiv_k m \iff k | m - n \]
1. Just like with $\equiv_3$, we'll more usually write: $n \equiv m \pmod{k}$ for $n, m \in \mathbb{Z}$.

2. Just like with $\equiv_3$, $\equiv_k$ is an equivalence relation for any $k \in \mathbb{Z}$.

---

**Nonexample of equiv. relations:**

1. Consider $\leq$ (e.g. on $\mathbb{R}$): is reflexive, transitive, but not symmetric, hence not an equiv. relation.

2. Consider the relation $\neq$ of inequality (e.g. on $\mathbb{Z}$)

   - Not symmetric, since $n \neq m \Rightarrow m \neq n$
   - Not reflexive (in fact: never true that $n \neq n$)
   - Not transitive (e.g. $2 \neq 4$ and $9 \neq 2$ but $2 = 2$)

---

**Equivalence Classes:** *Defn:* $[n]_R$ is the equivalence class of $x$, denoted $[x]_R$, is the set of elements related to $x$ by $R$. 

1. \( \{x \mid x \in E \} = \{y \in A \} (x, y) \in R^3 \)

(Note: by symmetry, could have as well defined \( \{x \mid x \in E \} = \{y \in A \} (y, x) \in R^3 \).)

Warning: overloaded notation. We've used E's when defining \( [n] = \{1, 2, \ldots, n\} \) — this is completely unrelated to meaning of \( \{x \mid x \in E \} \) for an equiv. relation \( R \) — so don't get confused!

\[ \text{Ex}^5: \] (a) \( aR = \{ x \in A \mid x = a \} \) denote the equality relation on \( N \). Then for any fixed \( a \in N \) we have:

\[ [a]_N = \{ m \in N \mid a = m \} \]

\[ = \{ n \} \]

e.g. \( [1]_N = \{1\}, [2]_N = \{2\}, \text{ etc.} \)

(b) \( aR \) denote the floor equiv. relation on \( \mathbb{R}^3 \): \( (x, y) \in R \) iff \( \|x\| = \|y\| \)

- Now! Fix \( x \in \mathbb{R}^3 \), and suppose \( \|x\| = n \).
- What is \( [x]_R \)?
by defn: \( C_x \mathcal{R}_p = \{ y \in \mathbb{R} \mid (x, y) \in \mathcal{R}_p \} \)

\[
= \{ y \in \mathbb{R} \mid \lfloor x \rfloor = \lfloor y \rfloor \}
= \{ y \in \mathbb{R} \mid n = \lfloor y \rfloor \}
= \{ y \in \mathbb{R} \mid n \leq y < n+1 \}
= \mathcal{C}_n \mathcal{C}_{n+1}
\]

Picture: e.g. if \( x = 2.34 \) so that \( \lfloor x \rfloor = 2 \)
we have \( \lceil x \rceil \mathcal{R}_p = \mathcal{C}_{2, 3} \)

\[
\begin{array}{cccc}
0 & 1 & 2 & 3 \\
\hline
0 & 1 & 2 & 3 \\
& \mathcal{C}_{\lfloor x \rfloor} \\
\end{array}
\]

Notice: the equiv. classes of \( \mathcal{R} \) form a partition of \( \mathbb{R} \):

\[
\begin{array}{cccc}
-1 & 0 & 1 & 2 \\
\hline
\mathcal{C}_{-1} & \mathcal{C}_0 & \mathcal{C}_1 & \mathcal{C}_2 \\
\end{array}
\]

we'll prove later: equivalence classes always partition the underlying set (in this case, \( \mathbb{R} \)).
Consider $\equiv_3$ on $\mathbb{Z}$. What are the
equivalence classes of this relation?

Let's write some down:

$[0]_{\equiv_3}$ is $\{n \in \mathbb{Z} | 0 \equiv_3 n \} = \{n \in \mathbb{Z} | 3 | n - 0 \} = \{n \in \mathbb{Z} | 3 | n \}$

$\vdots$

$[1]_{\equiv_3}$ is $\{n \in \mathbb{Z} | 1 \equiv_3 n \} = \{n \in \mathbb{Z} | 3 | n - 1 \}
= \{n \in \mathbb{Z} | (\exists k \in \mathbb{Z})(n-1 = 3k) \}
= \{n \in \mathbb{Z} | (\exists k \in \mathbb{Z})(n = 3k+1) \}
= \{ \ldots, -2, 1, 4, 7, \ldots \}$

$[2]_{\equiv_3}$ is $\{n \in \mathbb{Z} | 2 \equiv_3 n \}$

$\vdots$

$\vdots$

$\vdots$
\([3] \equiv_3 \Delta = \{ n \in \mathbb{Z} \mid 3 \mid n \} \]
\[= \{ n \in \mathbb{Z} \mid 3 \mid (n-3) \} \]
\[= \{ n \in \mathbb{Z} \mid 3 \mid n \} \]
\[= \{ ..., -3, 0, 3, 6, ... \} = [0] \equiv_3 \]

Similarly we can check:
\([4] \equiv_3 = [1] \equiv_3 \]
\([5] \equiv_3 = [2] \equiv_3 \]
\([6] \equiv_3 = [3] \equiv_3 = [0] \equiv_3 \), etc.

Notice: there are three distinct equivalence classes, each consisting of all integers of a given remainder when divided by 3 \((0, 1, \text{or} 2)\).

Again: the equivalence classes form a partition of \(\mathbb{Z}\) in this case.

\(\mathbb{Z} = \{ ..., -3, 0, 3, 6, ... \} \cup \{ ..., -2, 1, 4, 7, ... \} \)
\[\cup \{ ..., -1, 2, 5, 8, ... \} \]

Pairwise disjoint = \([0] \equiv_3 \cup [1] \equiv_3 \cup [2] \equiv_3 \)
Notation for congruence modulo \( k \)

we'll write \([ch]_k\) instead of \([ch]_k = k\).

e.g. \( \emptyset \) we'll abbreviate above as

\[ \emptyset = \{0\}_3 \cup \{1\}_3 \cup \{2\}_3 \]

Next goal is to see that "partition" and "equivalence relation" are, in a sense, two names for the same concept.

Recall: if \( A \) is a set, a partition \( \Pi \) of \( A \) is a set of subsets of \( A \) (i.e. \( \Pi \subseteq \mathcal{P}(A) \)) s.t.

1. \((\forall x \in \Pi) x \neq \emptyset\)
2. \((\forall x, y \in \Pi) (x \neq y \Rightarrow x \cap y = \emptyset)\)
3. \(\bigcup_{x \in \Pi} x = A\)

2 says the sets in \( \Pi \) are pairwise disjoint.

can also be written \((\forall x, y \in \Pi) (x = y \lor x \cap y = \emptyset)\)

Rec.

\( P = \{x, y, z\} \) is a partition of \( A \) (into 3 pieces)

\[
\begin{array}{ccc}
\text{A} & \text{x} & \text{y} \\
\text{z} & & \\
\end{array}
\]
Ex's: ① Let \( X = \{ \ldots, -3, 0, 3, 6, \ldots \} \)
\( Y = \{ \ldots, -2, 1, 4, 7, \ldots \} \)
\( Z = \{ \ldots, -1, 2, 5, 8, \ldots \} \)

Then \( P = \{ X, Y, Z \} \) is a partition of \( \mathbb{Z} \).

Proof: \( \emptyset \neq X, Y, Z \)
① \( X \cup Y = X \cup Z = Y \cup Z = \emptyset \)
② \( X \cap Y \cap Z = \emptyset \)

② For every \( n \in \mathbb{Z} \), define:
\( X_n = \{ y \in \mathbb{R} \mid n \leq y < n+1 \} \)
\( = \{ n, n+1 \} \)

Then \( P = \{ X_n \mid n \in \mathbb{Z} \} = \{ \ldots, X_{-1}, X_0, X_1, X_2, \ldots \} \)
is a partition of \( \mathbb{R} \).
Proof: you try

③ Let \( A = \{ 1, 2, 3, 47 \} = [47] \)

Define \( X = \{ 1 \} \) \( Y = \{ 2, 3, 47 \} \)

Then \( P = \{ X, Y \} = \{ \{ 1 \}, \{ 2, 3, 47 \} \} \)
is a partition of \( A \).
Equivalence classes partition sets.

Defn. Sups $R$ u an equiv relation on $A$. We define the set of equiv classes of $R$ as $A/R$

that u: $A/R = \{ [x]_R : x \in A \}$

read "A mod R".

Ex5: Consider $\equiv_3$ on $\mathbb{Z}$.

Then $\mathbb{Z}/\equiv_3 = \{ [n]_3 : n \in \mathbb{Z} \}$

$= \{ \ldots, [-3]_3, [0]_3, [3]_3, [6]_3, \ldots \}$

we already checked:

$= [3]_3 = [0]_3 = [3]_3 = [6]_3 = \ldots$

$= [2]_3 = [5]_3 = [8]_3 = \ldots$

$= [1]_3 = [4]_3 = [7]_3 = \ldots$

So really: $\mathbb{Z}/\equiv_3 = \{ [0]_3, [1]_3, [2]_3 \}$

which we could as well write:

$= \{ [6]_3, [-2]_3, [8]_3 \}$.