Ex: Q: For which \( n \in \mathbb{N} \) do we have \( n! > 2^n \)? Let's see…

\[
\begin{array}{c|c|c}
 n & n! & 2^n \\
\hline
 1 & 1 & 2 \\
 2 & 2 & 4 \\
 3 & 6 & 8 \\
 4 & 24 & 16 \\
 5 & 120 & 32 \\
\end{array}
\]

Seems like if \( n \geq 4 \) then \( n! > 2^n \). Let's prove.

Prop'n: For every \( n \in \mathbb{N} \) with \( n \geq 4 \) we have \( n! > 2^n \)

(here \( n_0 = 4 \) and \( S = \{ n \in \mathbb{Z} \mid n \geq 4 \} = \{ 4, 5, 6, \ldots \} \) )

Pf: Let \( P(n) \) be the prop'n "\( n! > 2^n \)".

(BC) \( P(4) \) holds since \( 4! > 2^4 \)

\[
2^4 = 16
\]

(\( \star \)) Fix \( n \in \mathbb{N} \), \( n \geq 4 \) [\( \text{NOTE: we fix } n \geq 4, \text{ not } n > 4 \) ] and assume \( P(n) \) holds

i.e. assume \( n! > 2^n \).
(15) Then we have:

\[(n+1)! = n! \cdot (n+1)\]

\[> 2^n \cdot (n+1) \quad (\text{by IH})\]

\[> 2^n \cdot 2 \quad (\text{since } n \geq 4, \quad n+1 \geq 5 > 2)\]

\[= 2^{n+1}\]

We've shown \((n+1)! > 2^{n+1}\), i.e. \(P(n+1)\) holds.

By induction, we've proved for every \(n \geq 4\) we have \(n! > 2^n\).

**Induction w/ Jumps**

-Sometimes we want to prove \(P(n)\), not for all \(n\), but when \(n\) is even, or... when \(n\) is odd, or... when \(n\) is a multiple of 3, etc.

We can still argue inductively.

Thus let \(P(n)\) be a var prop'n. Fix \(n_0 \in \mathbb{Z}\) and \(K \in \mathbb{N}\). (\(n_0 = \text{"starting point"}\))

Let \(S = \{n_0, n_0 + K, n_0 + 2K, \ldots\} \).
If we have

1. \( P(n) \)
2. \((\forall n \in \mathbb{N}) \ (P(n) \implies P(n+2)) \)

Then

\((\forall n \in \mathbb{N}) \ P(n) \) holds.

E.g. if \( S = \{2, 4, 6, \ldots \} = \mathbb{E} \) and we can show

1. \( P(2) \)
2. If \( P(n) \), then \( P(n+2) \)

then we've proved \( P(n) \) holds \( \forall n \in \mathbb{E} \).

Ex: Consider the alternating sum of the first \( n \) squares

\[ 1^2 - 2^2 + 3^2 - 4^2 + \ldots + (-1)^n n^2 \]

\[ = \sum_{k=1}^{n} (-1)^{k-1} k^2 \]

Prop'n 1. If \( n \) is odd we have:

\[ \sum_{k=1}^{n} (-1)^{k-1} k^2 = \sum_{k=1}^{n} k \ (\text{by before}) \]

Prop'n 2. If \( n \) is even, we have:

\[ \sum_{k=1}^{n} (-1)^{k-1} k^2 = -\sum_{k=1}^{n} k \]
Proof: where \( n_0 = 1 \) and jump = 2, so that 
\[ s = \{1, 3, 5, \ldots \} \]
\[ \sum_{k=1}^{n} (-1)^{k-1} k^2 = \sum_{k=1}^{n} k^n \]

(CBC) (For \( n = 1 \)) \[ \sum_{k=1}^{1} (-1)^{k-1} k^2 = 1^2 = 1 = \sum_{k=1}^{1} k \]

So \( P(1) \) holds.

(iii) Fix \( n \in \{1, 3, 5, \ldots \} \) and assume \( P(n) \),

1. Assume 
\[ \sum_{k=1}^{n} (-1)^{k-1} k^2 = \sum_{k=1}^{n} k^n \]

(ii) now consider the \( n+2 \) sum:
\[ \sum_{k=1}^{n+2} (-1)^{k-1} k^2 = \sum_{k=1}^{n} (-1)^{k-1} k^2 + (-1)^{n+1} (n+1)^2 + (-1)^{n+2} (n+2)^2 \]

\[ = \sum_{k=1}^{n} (-1)^{k-1} k^2 - (n+1)^2 + (n+2)^2 \]

\[ = \sum_{k=1}^{n} k^n - (n+1)^2 + (n+2)^2 \]

\[ = \sum_{k=1}^{n} k^n - (n+1)^2 + (n+2)^2 \]

\[ = \sum_{k=1}^{n} k^n + \frac{1}{2} \left[ (n+2)^2 - (n+1)^2 \right] \left[ (n+2) + (n+1) \right] \]

\[ = \sum_{k=1}^{n} k^n + (n+1) + (n+2) \]

\[ = \sum_{k=1}^{n} k^n + \sum_{k=1}^{n+2} k \]

\[ = \sum_{k=1}^{n+2} k \]

So \( P(n+2) \) holds.
By induction we've proved, and \( 1, 3, 5, \ldots \)
\[
\sum_{k=1}^{n} (-1)^{k-1} k^2 = \sum_{k=1}^{n} k
\]

**Summary:** we showed

1. \( P(1) \) holds
2. If \( n \in \{1, 3, 5, \ldots \} \) then \( P(n) \Rightarrow P(n+2) \)

It follows: \( P(n) \) holds \( \forall n \in \{1, 3, 5, \ldots \} \)

**2. For \( n \) even:**

\( \text{(BE)} \) \( \text{If } n=2 \):
\[
\sum_{k=1}^{2} (-1)^{k-1} k^2 = 1^2 - 2^2 = -3
\]
\[
= -\sum_{k=1}^{2} k
\]
\[
= -\sum_{k=1}^{2} k
\]

**IH** Fix \( n \in \{2, 4, 6, \ldots \} \) and assume:
\[
\sum_{k=1}^{n} (-1)^{k-1} k^2 = -\sum_{k=1}^{n} k
\]

**IS** Now, consider:
\[
\sum_{k=1}^{n+1} (-1)^{k-1} k^2 = \sum_{k=1}^{n} (-1)^{k-1} k^2 + (-1)^{n+1} (n+1)^2
\]
\[
= -\sum_{k=1}^{n} k + (n+1)^2 - (n+2)^2
\]
\[
= -\sum_{k=1}^{n} k + (n+1)^2 - (n+2)^2
\]
\[
\Rightarrow \text{since } n \text{ even}
\]
\[
= -\sum_{k=1}^{n} k - (n+1)\frac{1}{3} [(n+1)+(n+2)]
\]
\[
= -\sum_{k=1}^{n} k - [(n+1)+(n+2)]
\]
by induction, the identity holds \( \forall n \in \mathbb{Z}, n \geq 3 \)

**Fibonacci sequence**: is defined recursively by:

\[ F_0 = 0, \quad F_1 = 1 \]

\[ F_n = F_{n-2} + F_{n-1} \quad \text{for} \quad n > 2 \]

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, ...

\( F_0, F_1, F_2, F_3, F_4, \ldots \)

→ Fib sequence is a playground for inductive proofs.

**Prop’n** \( \forall n \in \mathbb{N}, n > 0 \) we have:

\[ \sum_{k=1}^{n} F_k = F_{n+2} - 1 \]

(i.e. \( F_0 + F_1 + \cdots + F_n = F_{n+2} - 1 \))

**Pf**: (B.C) if \( n = 1 \) we have:

\[ \sum_{k=1}^{1} F_k = F_1 = 1 = 2 - 1 = F_3 - 1 \]
(IH) Fix $n \in N$ and assume
\[ \sum_{k=1}^{n} F_k = F_{n+2} - 1 \]

(II) Consider:
\[ \sum_{k=1}^{n+1} F_k = \sum_{k=1}^{n} F_k + F_{n+1} \]
\[ \Rightarrow (F_{n+2} - 1) + F_{n+1} = F_{n+1} + F_{n+2} - 1 \]
\[ \Rightarrow \text{def. } f_{n+1} = F_{n+3} - 1 = F_{(n+1)+2} - 1 \]

by PMI: the $N \sum_{k=1}^{n} F_k = F_{n+2} - 1$ holds.

Prop/ If $n$ is a multiple of 3 (i.e. $n \in \{3, 6, \ldots\}$) then $F_n$ is even.

PF (BC) if $n=3$ then $F_3 = 2$ which is even.

(IH) Fix $n \in \{3, 6, 9, \ldots\}$ and assume $F_n$ is even.
(15) Consider \( F_{n+3} \):

\[
F_{n+3} = F_{n+2} + F_{n+1}
\]

\[
= \phi (F_{n+1} + F_n) + F_{n+1}
\]

\[
= 2F_{n+1} + F_n
\]

by the IH, \( F_n \) is even. Since \( 2F_{n+1} \) is even, \( 2F_{n+1} + F_n \) is even, i.e. \( F_{n+3} \) is even. 

By induction, \( 0, 1, 1, 2, 3, 5, 8, 13, 21, ... \) \( F_n \) is even.

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**Strong induction:**

- In certain proofs may need to assume more than \( P(n) \) to prove \( P(n+1) \)
- E.g. may need to assume \( P(n) \) and \( P(n-1) \) ... or even \( P(n), P(n-1), ..., P(1) \).
- Still a legit induction hypothesis!

**Thm** (Principle of strong mathematical induction PSMI)

Sps \( P(n) \) is a variable prop'n
If
1. \( P(i) \) holds
2. \((\forall n \in \mathbb{N}) (\forall k \in \mathbb{N}) P(k) \Rightarrow P(n+1) \) holds

Then \((\forall n \in \mathbb{N}) P(n) \) holds.

"\( \forall N \)"

Template For a PSMI Proof:
1. Prove \( P(i) \)
2. Fix \( n \in \mathbb{N} \). Assume \((\forall k \in \mathbb{N}) P(k) \) (i.e. assume \( P(1), P(2), \ldots, P(n) \))
3. Deduce \( P(n+1) \)

PSMI then gives: \((\forall n \in \mathbb{N}) P(n) \) holds.

Note: Despite name, PSMI seems weaker than PMI, because we have to assume more (namely all of \( P(1), P(2), \ldots, P(n) \) instead of just \( P(n) \)) to prove \( P(n+1) \).

- But, we'll later show PMI and PSMI are equivalent (and both equivalent to another principle called wcP).
Ex: Let $s_n$ be the sequence defined recursively by:

$$
\begin{align*}
S_0 &= 1 \\
S_n &= 1 + \sum_{k=0}^{n-1} S_k & \text{for } n \geq 1.
\end{align*}
$$

So e.g. $S_1 = 1 + S_0 = 1 + 1 = 2$

$S_2 = 1 + S_0 + S_1 = 1 + 1 + 2 = 4$

$S_3 = 1 + S_0 + S_1 + S_2 = 1 + 1 + 2 + 4 = 8$

It looks like $S_n = 2^n$

Let's prove this - we'll need a strong inductive hypothesis.

Prop'n true $\forall n \in \mathbb{N}_0$ we have $S_n = 2^n$.

**PF:** (BC) If $n = 0$, then $S_0 = 1 = 2^0 \checkmark$

(Strong IH) Fix $n \in \mathbb{N}_0$ and assume for every $k \in \{0, 1, \ldots, n\}$ we have $S_k = 2^k$.

(IS) now consider:
\[ S_{n+1} = 1 + \sum_{k=0}^{n} S_k \]
\[ = 1 + \sum_{k=0}^{n} 2^k \]

by strong IH

\[ = 1 + \frac{2^{n+1} - 1}{2 - 1} \]

by geometric series formula
proved before

\[ = 2^{n+1} \checkmark \]

\[ \Rightarrow \text{by PSMT, } S_n = 2^n \text{ for every } n \in \mathbb{N} \setminus \{0\}. \]

Notice: we really needed a strong IH since we need to replace every term in the sum \( \sum_{k} \) by \( 2^k \), not just the \( n \)-th term.

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**Def.** Given \( n \in \mathbb{N}, n>1 \), a **prime factorization** of \( n \) is a way of writing \( n \) as a product of primes.