Induction:

Q: what happens if we add the first $n$ odd positive integers together?

\[
\begin{align*}
1 &= 1 = 1^2 \\
1 + 3 &= 4 = 2^2 \\
1 + 3 + 5 &= 9 = 3^2 \\
1 + 3 + 5 + 7 &= 16 = 4^2 \\
(*): 1 + 3 + 5 + \ldots + (2n-1)^2 &= n^2
\end{align*}
\]

Picture

\[n^2 + (2n+1) = (n+1)^2\]
- The picture suggests $\text{(1)}$ is true for all $n \in \mathbb{N}$. How would we prove it?

- Pic also suggests that proof for $n+1$ depends on proof for $n$.

**Theorem**: For every $n \in \mathbb{N}$, we have:

$$1 + 3 + 5 + \ldots + (2n-1) = n^2$$

1. i.e. $\sum_{k=1}^{n} 2k-1 = n^2$

**Proof**: Clearly true when $n=1$ since:

$$\sum_{k=1}^{1} 2k-1 = 1 = 1^2$$

- Suppose that $n \in \mathbb{N}$ is fixed and we have that the identity holds for $n$.

  i.e. assume that:

  $$\sum_{k=1}^{n} 2k-1 = n^2$$

- Now consider the sum for $n+1$:

  $$\sum_{k=1}^{n+1} 2k-1 = 1 + 3 + \ldots + (2n-1) + 2(n+1)-1$$

  $\sum_{k=1}^{n} 2k-1$
\[ \sum_{k=1}^{n} 2k - 1 + 2n+1 = n^2 + 2n + 1 \]

by our assumption

we've shown:

(a) identity holds for \( n = 1 \)
(b) if it holds for a fixed \( n \in \mathbb{N} \),
then it also holds for \( n+1 \).

It follows: since identity holds for \( n = 1 \)
it holds for \( n = 2 \),
and so also for \( n = 3 \),
and so also for \( n = 4 \),
and so for all \( n \in \mathbb{N} \)!

The validity of this kind of argument is called the principle of mathematical induction (PMI)
"Theorem" (PMI) Suppose \( P(n) \) is a variable proposition. Suppose further that:

1. \( P(1) \) holds
2. \( (\forall n \in \mathbb{N}) (P(n) \Rightarrow P(n+1)) \) holds

then:

\( (\forall n \in \mathbb{N}) (P(n)) \) holds.

1. For a "proof" see the book.
2. We'll take PMI as an axiom (i.e., we'll assume the type of reasoning we used above is valid).
3. Later, we'll show PMI is equivalent to another intuitively obvious principle.

Using PMI to prove: \( (\forall n \in \mathbb{N}) P(n) \)

1. (Base case) Verify \( P(1) \) directly
2. (Induction hypothesis) Fix \( n \in \mathbb{N} \) and assume \( P(n) \) holds
3. (Induction step) Using the hypothesis deduce \( P(n+1) \).
PMI says: if you can do 0, 2, 3 then \( (\forall n \in \mathbb{N}) P(n) \) holds.

**Ex:** What happens if we sum the first \( n \) natural numbers?

\[
1 + 2 + \ldots + n = ?
\]

**First few:**

<table>
<thead>
<tr>
<th>( n )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 1 + 2 )</td>
<td>3</td>
<td>6</td>
<td>10</td>
<td>( \frac{n(n+1)}{2} )</td>
</tr>
<tr>
<td>( 1 + 2 + 3 )</td>
<td>6</td>
<td>10</td>
<td>( \frac{n(n+1)}{2} )</td>
<td></td>
</tr>
<tr>
<td>( 1 + 2 + 3 + 4 )</td>
<td>10</td>
<td>( \frac{n(n+1)}{2} )</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

\[ \Rightarrow 1 + 2 + \ldots + n = \frac{n(n+1)}{2} \]

**Theorem:** For every \( n \in \mathbb{N} \) we have:

\[
\sum_{k=1}^{n} k = \frac{n(n+1)}{2}
\]

**PF:** Let \( P(n) \) be the prop'nh

\[
\sum_{k=1}^{n} k = \frac{n(n+1)}{2}
\]

**BC:** \( P(1) \) is true since

\[
\sum_{k=1}^{1} k = 1 = \frac{1 \cdot 2}{2}
\]
**Mathematical Induction (IH):** Fix $n \in \mathbb{N}$, and assume $P(n)$, i.e., assume

\[ \sum_{k=1}^{n} k = \frac{n(n+1)}{2} \]

**Now consider:**

\[ \sum_{k=1}^{n+1} k = 1 + 2 + \ldots + n + (n+1) \]

\[ = \sum_{k=1}^{n} k + (n+1) \]

by 

\[ = \frac{n(n+1)}{2} + (n+1) \]

by IH

\[ = \frac{n(n+1)}{2} + \frac{2(n+1)}{2} \]

\[ = \frac{n(n+1) + 2(n+1)}{2} \]

\[ = \frac{(n+1)(n+2)}{2} \]

\[ = \frac{(n+1)(n+1+1)}{2} \]

**hence $P(n+1)$ holds.**

**by PMI, $P(n)$ holds for every $n \in \mathbb{N}$**

i.e.

\[ \sum_{k=1}^{n} k = \frac{n(n+1)}{2} \]
Notice: proof doesn't really give insight into how we might have guessed the formula:
\[ \sum_{k=1}^{n} k = \frac{n(n+1)}{2} \]

But: once we have guessed formula, PMT gives us a way of verifying it's really true. Then.

2. Geometric Series) Fix \( x \in \mathbb{R} \) with \( x \neq 0, 1 \). Then for every \( n \in \mathbb{N} \) we have:
\[ 1 + x + x^2 + \ldots + x^{n-1} = \frac{x^n - 1}{x - 1} \]

\[ \sum_{k=0}^{n-1} x^k = \frac{x^n - 1}{x - 1} \]

Call this \( P(n) \).

(B0) \( P(1) \) holds since:
\[ \sum_{k=0}^{0} x^k = x^0 = 1 = \frac{x^1 - 1}{x - 1} \]

since \( x \neq 0 \) since \( x \neq 1 \)
(IH) Fix \( n \in \mathbb{N} \) and assume \( P(n) \), i.e. assume \( \sum_{k=0}^{n-1} x^k = \frac{x^n - 1}{x - 1} \)

(IS) Now consider:
\[
\sum_{k=0}^{n} x^k = \sum_{k=0}^{n-1} x^k + x^n
\]

by IH:
\[
\sum_{k=0}^{n-1} x^k = \frac{x^n - 1}{x - 1} + x^n
\]

\[
= \frac{x^n - 1}{x - 1} + \frac{x^n(x - 1)}{x - 1}
\]

\[
= \frac{x^n(x - 1) + x^n - x^n}{x - 1}
\]

\[
= \frac{x^{n+1} - 1}{x - 1}
\]

\[
\Rightarrow P(n+1) \text{ holds.}
\]

By PMI, \( P(n) \) holds for all \( n \in \mathbb{N} \)

i.e. for all \( n \in \mathbb{N} \) we have \[
\sum_{k=0}^{n-1} x^k = \frac{x^n - 1}{x - 1}
\]
Prop'n For every $n \in \mathbb{N}$, $7^n - 4^n$ is a multiple of 3.

**PF** (Pc) If $n = 1$, statement holds since $7^1 - 4^1 = 3$.

(IH) Fix $n \in \mathbb{N}$ and assume for all $k \leq n$ s.t. $7^k - 4^k = 3k$ ($k \in \mathbb{N}$)

(IS) Now, observe:

$$7^n = 3k + 4^n \quad \text{(by IH)}$$

$$\Rightarrow 7^{n+1} = (3k + 4^n) \cdot 7$$

$$= 21k + 7 \cdot 4^n$$

$$= 21k + (3 + 4^n) \cdot 4^n$$

$$= 21k + 3 \cdot 4^n + 4^{n+1}$$

$$\Rightarrow 7^{n+1} - 4^{n+1} = 21k + 3 \cdot 4^n$$

$$= 3(7k + 4^n) = 3M$$

where $M \geq 7k + 4^n$

hence $7^{n+1} - 4^{n+1}$ is a multiple of 3.

By PMI, $7^n - 4^n$ is a multiple of 3 for every $n \in \mathbb{N}$.
Variants of Induction

- Nothing special about \( n = 1 \) as a base case.

Then (PMI w/ a different BC)
- Spe P(\( n \)) is a var. prop'n and \( n_0 \in \mathbb{Z} \).
  - Is fixed (possibly negative)
- Let \( S = \{ n_0, n_0 + 1, n_0 + 2, \ldots \} = \{ n \in \mathbb{Z} | n \geq n_0 \} \)

If we have
  1. P(\( n_0 \)) holds
  2. \((\forall n \in S) (P(n) \Rightarrow P(n+1)) \) holds

Then
  \( (\forall n \in S) \) P(\( n \)) holds

\( \rightarrow \) can prove theorem using regular PMI (see book)

\( \rightarrow \) proof template nearly the same as usual PMI

\( \circ \) (BC) Verify P(\( n_0 \))
\( \circ \) (IH) Fix n \epsilon S. Assume P(n)
  - I.e. \( n \in \mathbb{Z} \)
  - \( n \geq n_0 \)
\( \circ \) (TS) Prove P(n+1)