more precise to observe that statements \("x \in A \land B\)" etc.
"x \in A \lor B"
are equivalent to
"(x \in A) \land (x \in B)" etc.
"(x \in A) \lor (x \in B)" etc.

We can use this observation to prove the equality of two sets in a new way, using \(U\)’s
Theorem Suppose \(A, B\) are sets and \(U\) is a universal set with \(A \subseteq U\).

Then we have:
1. \(\overline{A} = A\)
2. \(A \cap B = \overline{A} \cup \overline{B}\)
3. \(A \cup B = \overline{A} \cap \overline{B}\)

**PF:** \(\exists x \in U \leftarrow \text{not in } A \text{ or } \overline{A} \text{ !!!!}
\)
then \(x \in \overline{A} \Leftrightarrow x \notin \overline{A}\) (def'n of complement)
\(\Leftrightarrow \neg (x \in \overline{A})\)
\(\Leftrightarrow \neg (\neg (x \in A))\) (def'n of comp. ^{\text{w}})
\(\Leftrightarrow x \in A\)
This chain of equivalences shows:
$(x \in \bar{A}) \equiv (x \in A)$

1.e. $(x \in \bar{A}) \Rightarrow (x \in A)$ ← this shows $\bar{A} \subseteq A$
and $(x \in A) \Rightarrow (x \in \bar{A})$ ← $A \subseteq \bar{A}$

hence we've proved $\bar{A} = A$.

2. Fix $x \in U$
then $x \in A \land B$ $\Leftrightarrow$ $\neg \neg (x \in A \land B)$
   $\Leftrightarrow \neg \neg ((x \in A) \land (x \in \bar{B}))$
   $\Leftrightarrow \neg \neg \neg (x \in A) \land \neg \neg (x \in \bar{B})$
   $\Leftrightarrow (x \in A) \lor (x \in \bar{B})$
   $\Leftrightarrow (x \in A) \lor (x \in \bar{B})$
   $\Leftrightarrow x \in A \lor \bar{B}$

This proves: $A \land B = \bar{A} \lor \bar{B}$


Theorem: For any sets $A, B, C$ we have:

1. $A \land (B \lor C) = (A \land B) \lor (A \land C)$
2. $A \lor (B \land C) = (A \lor B) \land (A \lor C)$

Proof: you try (use the logical distributive laws)
Proof Writing:
Always two approaches: when trying to prove statement $P$, can either prove directly, or assume $\neg P$ and derive a contradiction.

More generally: can prove any statement logically equivalent to $P$ or disprove any statement logically equivalent to $\neg P$.

Existence Claims
General Form: $(\exists x \in S) P(x)$

Direct proof strategy: define a specific $y \in S$ and show $P(y)$ holds.

Example: There exists an even number $n \in \mathbb{N}$ that can be written as the sum of two primes in two distinct ways.

Proof: Consider $n = 10$. Then $10$ is even and we have: $10 = 5 + 5 = 7 + 3$.

Since $3, 5, 7$ are prime, the claim is proved.
Indirect Proof Strategy: Assume \( \neg (\exists x \in S) P(x) \) (equiv: \( \forall x \in S \neg P(x) \)) and derive a contradiction.

**Ex:** Fix \( n \in \mathbb{N} \) and \( a_1, \ldots, a_n \), an \( S \subset \mathbb{R} \).

Then there is a \( k \in \{1, 2, \ldots, n\} \) s.t. \( a_k \) is at least as large as the average (mean) of \( a_1, \ldots, a_n \).

That is:

\[
(\exists k \in \{1, n\}) \left( a_k \geq \frac{1}{n} (a_1 + a_2 + \ldots + a_n) \right)
\]

**PF:** Suppose not, toward a contradiction.

That is, suppose

\[
(\forall k \in \{1, n\}) \left( a_k < \frac{1}{n} (a_1 + \ldots + a_n) \right)
\]

For simplicity, let \( s = a_1 + a_2 + \ldots + a_n \).

So our assumption is:

\[
(\forall k \in \{1, n\}) \left( a_k < \frac{s}{n} \right)
\]

But then:

\[
s = a_1 + a_2 + \ldots + a_n < \frac{s}{n} + \frac{s}{n} + \ldots + \frac{s}{n} \text{ (defn of } s)\]

(by our assumption)

\[
\frac{s}{n} \times n = s \text{ (n times)}
\]
\[
= n \cdot \frac{S}{n} = S
\]

This shows \( S > S \), a contradiction.
Thus our assumption was false.
Hence the prop'n is true.

**Universal Claims**

**General Form:** \((\forall x \in S) P(x)\)

**Direct Strategy:** Let \( x \in S \) be arbitrary but fixed.

- Prove \( P(x) \) holds.

**Ex 6 Prop'n** \((\forall x, y \in \mathbb{R}) (xy \leq \left(\frac{x+y}{2}\right)^2)\)

**PF:** Fix \( x, y \in \mathbb{R} \).

- Then: \((x-y)^2 \geq 0\) (squares always \( \geq 0 \)).
- Hence \( x^2 - 2xy + y^2 \geq 0 \).

\[\Rightarrow x^2 + y^2 \geq 2xy\]  (adding \( 2xy \) to both sides)

\[\Rightarrow x^2 + 2xy + y^2 \geq 4xy\]

i.e. \((x+y)^2 \geq 4xy\)

\[\Rightarrow \left(\frac{x+y}{2}\right)^2 \geq xy\]

\[\Rightarrow \left(\frac{x+y}{2}\right)^2 \geq xy\] , as desired.
Since \( x, y \in \mathbb{R} \) was arbitrary, the claim is proved.

Aside: prop'n is one version of the AM-GM inequality:

- Arithmetic Mean (AM) of \( x, y \) is \( \frac{x+y}{2} \)
- Geometric Mean (GM) of \( x, y \) is \( \sqrt{xy} \)

\[ \frac{x+y}{2} \]

\[ \frac{\sqrt{xy}}{2} \]

\[ \frac{x}{y} \quad \frac{y}{x} \]

\[ \sqrt{xy} \]

Prop'n proves (for \( x, y \geq 0 \)) that

\[ \sqrt{xy} \leq \frac{x+y}{2} \]

i.e. \( GM \leq AM \).

Indirect Strategy: Assume \( \exists ( \forall x : P(x) \land (\exists x : P(x))) \) and derive a contradiction.
Ex 2 \( \sqrt{2} \) is irrational, that is, 
\((a,b \in \mathbb{Z}) \ (\frac{a}{b} \neq \sqrt{2})\)

PF: - Suppose, that is, suppose \( a,b \in \mathbb{Z} \) such that \( \frac{a}{b} = \sqrt{2} \)

- We may assume \( \frac{a}{b} \) in reduced form, i.e. that \( a,b \) share no common factors; if they did, we could cancel these factors to get \( a',b' \in \mathbb{Z} \) with no common factors, such that \( \frac{a'}{b'} = \sqrt{2} \) and is reduced.

- Now, since \( \frac{a}{b} = \sqrt{2} \)
  
  we have \( a = \sqrt{2} b \)
  
  \( \Rightarrow a^2 = 2b^2 \)
  
  - Hence \( a^2 \) is even. It follows \( a \) itself is even (why?)

- Hence \( \exists k \in \mathbb{Z} \) s.t. \( a = 2k \)

- So then \( a^2 = 4k^2 \)

- Which gives \( 2b^2 = 4k^2 \)
which gives: \( b^2 = 2k^2 \)
- reasoning as before we see that \( b^2 \) and hence \( b \) is even.
- so both \( a \) and \( b \) are even: hence they share a factor of 2.
- a contradiction, as \( a \) and \( b \) share no common factors!
- the proof follows.

**Conditional Claims**

**General Form:** \( P \Rightarrow Q \)

**Three Strategies:**

1. **Direct:** Assume \( P \) holds, prove \( Q \).
2. **Contrapositive:** Prove \( \neg Q \Rightarrow \neg P \), i.e., assume \( \neg Q \) and prove \( \neg P \).
3. **Indirect:** Assume \( \neg (P \Rightarrow Q) \) (equiv: \( P \land \neg Q \)) and derive a contradiction.

**Exercise (Direct):** Let \( \mathcal{O} = \{0, 1, 2, ..., -3, -1, 1, 3, ...\} \) denote the set of all odd integers

(1) negative.)
Propn: \((\forall n \in \mathbb{Z})(n \neq 0 \Rightarrow n^2 - 1 \text{ is divisible by } 4)\)

Or: even more symbolically
\((\forall n \in \mathbb{Z})(n \neq 0 \Rightarrow \exists k \in \mathbb{Z} \text{ s.t. } n^2 - 1 = 4k)\)

Pf: [Overall: this is a universal claim, so we begin as usual]

- Fix \(n \in \mathbb{Z}\)
- Now we deal with the condition:
- Assume \(n \neq 0\)
- We're allowed to do this because if \(n \neq 0\), the conditional claim holds vacuously.
- Then \(\exists k \in \mathbb{Z}\) s.t. \(n = 2k + 1\)
- Hence \(n^2 = (2k+1)^2 = 4k^2 + 4k + 1\)
- \(\Rightarrow n^2 - 1 = 4k^2 + 4k\)
- \(\Rightarrow n^2 - 1 = 4(k^2 + k) = 4M\) (where \(M = k^2 + k\))
- Hence \(n^2 - 1\) is divisible by 4
- Since \(n\) was arbitrary, the claim is proved. \(\checkmark\)
The set of all even integers is denoted as $E = \{ \ldots, -4, -2, 0, 2, 4, \ldots \}$.

Prop: $(\forall m, n \in \mathbb{Z}) \ (mn \in E \Rightarrow (m \in E) \lor (n \in E))$

**Pf.** - Fix $m, n \in \mathbb{Z}$. We argue the contrapositive by contradiction.

- Assume $(m \notin E \lor n \notin E)$, i.e., $m \notin E \land n \notin E$

- Then $m, n$ are both odd.

- Hence $3k, 6l + 1$ s.t.
  
  
  
  $m = 2k + 1$
  
  $n = 2l + 1$

- Then $mn = (2k+1)(2l+1)$

  $= 4kl + 2k + 2l + 1$

  $= 2(2kl + k + l) + 1$

  $= 2M + 1$ (where $M = 2kl + k + l$)

- Hence $mn \notin \mathbb{E}$, i.e., $mn \notin E$.

- We've proved $(m \notin E \land n \notin E) \Rightarrow mn \notin E$

  i.e., $(\neg (m \in E) \lor \neg (n \in E)) \Rightarrow \neg (mn \in E)$
- by contrapositive, we've proved
  \[ m \neq E \implies m \in E \lor m \notin E \]
- since \( m, n \in \mathbb{Z} \) were arbitrary, the claim is proved.

(3) (Indirect) Prove: \((\forall x \in \mathbb{R}) \ (x > 0 \implies x + \frac{1}{x} \geq 2)\)

PF: - fix \( x \in \mathbb{R} \)
  - suppose \( x > 0 \) but \( x + \frac{1}{x} < 2 \)
  \[ \implies x^2 + 1 < 2x \] (inequality doesn't flip since \( x > 0 \))
  \[ \implies x^2 - 2x + 1 < 0 \]
  \[ \implies (x - 1)^2 < 0 \]
  a contradiction, as the quantity \((x - 1)^2 \geq 0\).
  - hence we must have
    \[ x > 0 \implies x + \frac{1}{x} \geq 2 \]
  - since \( x \) was arbitrary, the claim is proved.
Bi-conditional Claims

**General Form:** \( P \iff Q \)

**Strategy:** Prove \( P \implies Q \) and \( Q \implies P \).

**Ex:** Prop'n, An integer \( n \) is even if and only if its square is even.

i.e.

\[
(\forall n \in \mathbb{Z}) (n \in \mathbb{E} \iff n^2 \in \mathbb{E})
\]

**PF:** Fix \( n \in \mathbb{Z} \)

\((\Rightarrow)\) Assume \( n \in \mathbb{E} \)

- Then \( \exists k \in \mathbb{Z} \) s.t. \( n = 2k \)
- Hence \( n^2 = (2k)^2 = 4k^2 \)
  
  \( = 2(2k^2) \)
  
  \( = 2M \) (where \( M = 2k^2 \))

- Hence \( n^2 \) is even i.e. \( n^2 \in \mathbb{E} \).

\((\Leftarrow)\) To prove \( n^2 \in \mathbb{E} \implies n \in \mathbb{E} \) we show the contrapositive: \( n \notin \mathbb{E} \Rightarrow n^2 \notin \mathbb{E} \)
- So suppose \( n \notin \mathbb{E} \)
- then \( n \) is odd, i.e. \( \exists k \in \mathbb{Z} \) s.t. \( n = 2k + 1 \)
- hence \( n^2 = (2k+1)^2 \)
  \[ = 4k^2 + 4k + 1 \]
  \[ = 2(2k^2 + 2k) + 1 \]
  \[ = 2M + 1 \quad (M = k^2 + 2k) \]
- hence \( n^2 \) is odd, i.e. \( n^2 \notin \mathbb{E} \).
- by contrapositive we've proved \( n^2 \notin \mathbb{E} \iff n \notin \mathbb{E} \).
- hence \( n \notin \mathbb{E} \iff n^2 \in \mathbb{E} \).
- since \( n \) was arbitrary, the prop'n is proved.