- can also use connectives in def'n, set-builder notation, etc.
- e.g. if \( A, B \) are subsets of a universal set \( U \), can define:
  \[
  A \cap B = \{ x \in U \mid (x \in A) \land (x \in B) \} \\
  A \cup B = \{ x \in U \mid (x \in A) \lor (x \in B) \} \\
  \complement A = \{ x \in U \mid \neg (x \in A) \}
  \]
equiv to "\( x \notin A \)"

- we'll explore connections between connectives and set operations more later.

**Implication**: Given statements \( P, Q \)
the statement \( P \implies Q \) is read "if \( P \), then \( Q \)" or "\( P \) implies \( Q \)"
- \( P \implies Q \) is true iff
when \( P \) is true, \( Q \) is also true.

<table>
<thead>
<tr>
<th>( P )</th>
<th>( Q )</th>
<th>( P \implies Q )</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>T</td>
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<tr>
<td>F</td>
<td>F</td>
<td>T</td>
</tr>
</tbody>
</table>

Notice: \( \neg P \implies Q \) is always true when \( P \) is false (often a confusing point)
- \( P \implies Q \) is only false when \( P \) is true and \( Q \) is false.
Statements of the form \( P \Rightarrow Q \) are called **conditional statements**.

**Ex 5**

1. "\( (1+1=2) \Rightarrow (1+1+1=3) \)" is **true**
   \[ \frac{\text{true}}{\Rightarrow} \]

2. "\( (1+1=2) \Rightarrow (1+1+1=4) \)" is **false**
   \[ \frac{\text{true}}{\Rightarrow} \]

3. "\( (1+1=2) \Rightarrow (1^2 \notin N) \)" is **true**
   \[ \frac{\text{true}}{\Rightarrow} \]

Even though \( P \) and \( Q \) in this ex are not apparently related statements.

4. "My name is Sally \( \Rightarrow \) My name begins with \( S \)" is **true**

   \( \Rightarrow \) both the premise \( P \) and conclusion \( Q \) are **false**), but (by def'n) therefore \( P \Rightarrow Q \) is **true**

   \( \Rightarrow \) illustrates why "false \( \Rightarrow \) false" is \( \text{true} \)

5. "Tomorrow \( \Rightarrow \) Sunday \( \Rightarrow \) my name is Garrett" is also **true**

   ("false \( \Rightarrow \) true" is **true**).
6. \((\exists x \in \mathbb{R})(x^2 = -1) \Rightarrow (1 + 1 = 3)\) is **true**! Automatically since premise is false even though it's unrelated to conclusion.

7. Can also use \(\Rightarrow \) in vac. prop'w l.g.

\[ x \geq 2 \Rightarrow x^2 \geq 4 \]

is a well-formed vac prop'n and

\((\forall x \in \mathbb{R})(x \geq 2 \Rightarrow x^2 \geq 4)\)

is **true**, because:

**for every** \(x \in \mathbb{R}\), either \(x \geq 2\), in which case \(x^2 \geq 4\). Hence \(x \geq 2 \Rightarrow x^2 \geq 4\) for such \(x\), since "true \(\Rightarrow\) true" \(\Rightarrow\) true.

or \(x < 2\), in which case \(x^2 < 4\) \(\Rightarrow\) false \(\Rightarrow\) true, since "false \(\Rightarrow\) false" \(\Rightarrow\) true.

\(\Rightarrow\) for every \(x \in \mathbb{R}\), "\((x \geq 2 \Rightarrow x^2 \geq 4)\)" \(\Rightarrow\) (T)

i.e. "\((\forall x \in \mathbb{R})(x \geq 2 \Rightarrow x^2 \geq 4)\)" is (T) as claimed.

5. **CTOH**: \((\forall x \in \mathbb{R})(x^2 \geq 4 \Rightarrow x \geq 2)\) is **false** because: there is a real number \(x\) (e.g. \(x = -3\)) s.t. "\(x^2 \geq 4\)" \(\Rightarrow\) (T) but "\(x \geq 2\)" \(\Rightarrow\) (F)

i.e. there \(x \in \mathbb{R}\) s.t. "\((x^2 \geq 4) \Rightarrow (x \geq 2)\)" \(\Rightarrow\) (F)
Equivalence: Given statements $P \land Q$ (14)
the statement $P \Leftrightarrow Q$
(read: "P if and only if Q" or "P IFF Q")
is true iff $P, Q$ have the same truth value.

<table>
<thead>
<tr>
<th>$P$</th>
<th>$Q$</th>
<th>$P \Leftrightarrow Q$</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>T</td>
</tr>
</tbody>
</table>

Ex.
1. $1+1=2 \Leftrightarrow (1+1=3) \vee (1)$
2. $1+1=3 \Leftrightarrow (1+1=4) \vee (1)$
3. $(\forall x \in \mathbb{N}) (x > 0) \Leftrightarrow (1+1=2) \vee (1)$

$P, Q$ need not be "related."
4. $1+1=2 \Leftrightarrow (2+2=5) \vee (1)$

- can also use $\Leftrightarrow$ in var. prop'n, e.g.

$\left( x > 0 \right) \Leftrightarrow \left( \exists y \in \mathbb{R} \right) \left( x = y^2 \right)$

is a legit var. prop'n, and the statement

$\left( \forall x \in \mathbb{R} \right) \left[ \left( x > 0 \right) \Leftrightarrow \left( \exists y \in \mathbb{R} \right) \left( x = y^2 \right) \right]$ is True:

- why: For any fixed $x \in \mathbb{R}$, the statements "$x > 0$" and "$\exists y \in \mathbb{R} \left( x = y^2 \right)$" are either both true, or both false.
Def'n Two statements \( P, Q \) are said to be logically equivalent iff they have the same truth value, i.e. iff \( P \equiv Q \) is true.

- e.g. \( 1 + 1 = 2 \) and \( 1 + 1 + 1 = 3 \) are logically equivalent.

- we're most interested in logically equivalent forms for connected (e.g. negated) and quantified statements.

Negating Quantified Statements

- Sp: \( P(x) \) is a var prop'n and \( S \) a set.
- Consider the negated statements:
  1. \( \neg \exists x \in S \ P(x) \)
  2. \( \neg \forall x \in S \ P(x) \)

- Observe: 1. is true iff there \( u \) an \( x \in S \) s.t. \( P(x) \) is false, i.e. iff \( \exists x \in S \ P(x) \) is true.

  2. is true iff for all \( x \in S \) we have that \( P(x) \) is false, i.e. iff \( \forall x \in S \ P(x) \) is true.
This shows:

$$
\lnot (\forall x \in S) P(x) \iff (\exists x \in S) \lnot P(x)
$$

is always true (regardless of the property P(x))

i.e. that \( \lnot (\forall x \in S) P(x) \) and \( (\exists x \in S) \lnot P(x) \) are logically equiv.

- likewise \( \lnot (\exists x \in S) P(x) \) and \( (\forall x \in S) \lnot P(x) \) are logically equiv.

- these equivalences often useful when trying to prove quantified statements by contradiction.

Ex's: 0. \( \lnot (\forall x \in \mathbb{R}) (x \in \mathbb{N}) \) “not all reals are naturals” is equiv to

\( (\exists x \in \mathbb{R}) \lnot (x \in \mathbb{N}) \) “there is a real which is not a natural”

(\textit{note}: we'll often write \( \lnot (x \in \mathbb{N}) \) as \( x \notin \mathbb{N} \), \( \lnot (x = y) \) as \( x \neq y \), etc.)

2. \( \lnot (\exists x \in \mathbb{R}) (x+1 = 0) \) “there is no additive inverse for \( 1 \) in \( \mathbb{R} \)” is equiv to

\( (\forall x \in \mathbb{R}) (x+1 \neq 0) \) “every real \( \neq \) 0 has an additive inverse in \( \mathbb{R} \)”
In this case, both statements are false.

3. For multiple quantifiers: iterate the process.

\( \forall x \in \mathbb{R} \exists y \in \mathbb{R} (xy = 1) \)

equivalent to:

\( \exists x \in \mathbb{R} \forall y \in \mathbb{R} (xy = 1) \)

equivalent to:

\( \forall x \in \mathbb{R} \exists y \in \mathbb{R} (xy \neq 1) \)

(These statements are true:

\( \exists a \) has no multiplicative inverse)

Negating connected statements:

Proposition For any statements \( P, Q \), the following logical equivalencies hold:

1. \( \neg \neg P \equiv P \)
2. \( \neg (P \land Q) \equiv \neg P \lor \neg Q \) "De Morgan's Laws"
3. \( \neg (P \lor Q) \equiv \neg P \land \neg Q \)

Proof: To prove, we'll use truth tables:

<table>
<thead>
<tr>
<th>( P )</th>
<th>( \neg P )</th>
<th>( \neg \neg P )</th>
<th>( \neg \neg P \equiv P )</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>F</td>
<td>T</td>
<td>( \checkmark )</td>
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<tr>
<td>F</td>
<td>T</td>
<td>F</td>
<td>( \checkmark )</td>
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</table>

\( \neg P \equiv P \) is always true when \( P \) is false.
Ex: ① \((1 \cdot 1 = 2)\) 
Equivalent: \(1 + 1 = 2\) (both \(T\))
② \(7 ((1 + 1 = 2) \land (1 + 1 = 3))\)
Equivalent to \((1 + 1 \neq 2) \lor (1 + 1 \neq 3)\) (both \(T\))
③ \(7 ((1 + 1 = 2) \lor (1 + 1 = 3))\) (both \(F\))
Equivalent: \((1 + 1 \neq 2) \land (1 + 1 \neq 3)\)
④ \((\forall x \in \mathbb{R}) \neg ((x < 0 \land (\exists y \in \mathbb{R})(y^2 = x)))\)
⑤ \((\forall x \in \mathbb{R}) \neg ((x < 0) \lor \neg (\exists y \in \mathbb{R})(y^2 = x))\)
(all \(true\))

Equivalences for \(\Rightarrow\): Proven For any \(P \land Q\) the following equivalences hold:
① \((P \Rightarrow Q) \Leftrightarrow (\neg P \lor Q)\)
② \((P \Rightarrow Q) \Leftrightarrow (\neg Q \Rightarrow \neg P)\)
③ \((P \Leftrightarrow Q) \Leftrightarrow (P \Rightarrow Q \land Q \Rightarrow P)\)
Proof of (i) + (ii):

<table>
<thead>
<tr>
<th>P</th>
<th>Q</th>
<th>P → Q</th>
<th>T P</th>
<th>T Q</th>
<th>T P ∨ T Q</th>
<th>T Q → T P</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
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</table>

\[(P → Q) ↔ (T P ∨ T Q) \quad (P → Q) ↔ (T Q → T P)\]

Proof of (iii): you try.

Note: these equivalences are very useful for proving statements of the form \( P → Q \) and \( P ↔ Q \).

Negating ever → and ↔ Prop's: the following logical equivalences hold.
1. \( T(P \Rightarrow Q) \equiv (P \land \neg Q) \)
2. \( T(P \Rightarrow Q) \equiv [(P \land \neg Q) \lor (\neg P \land Q)] \)

Proof: you try.

Note: with these and our previous equivalences, we can now put any negated statement into "positive form".

Def'n A statement \( P \) is in positive form iff any negation symbols in \( P \) only occur next to substatements that contain no connecting or quantifiers (i.e. negation symbols are "as inside as possible").

Our rules above enable us to find, for any \( P \), a logically equivalent statement \( P' \) in positive form:

\[
\begin{align*}
\text{Ex's} & \quad (5 \in \mathbb{E}) \Rightarrow (6 \in \mathbb{E}) \quad \text{is equiv} \\
& \quad \text{to:} \quad (5 \in \mathbb{E}) \lor (6 \in \mathbb{E}) \\
& \quad \text{which we can write:} \quad (\neg 6 \in \mathbb{E}) \lor (6 \in \mathbb{E}) \quad (T)
\end{align*}
\]
2. \((\forall x \in \mathbb{N}) (x \in D) \Rightarrow (x+1 \in E)\)
   
   equiv to:
   
   \((\forall x \in \mathbb{N}) ((x \notin C) \lor (x+1 \in E))\)
   
   also equiv to:
   
   \((\forall x \in \mathbb{N}) ((x+1 \notin E) \Rightarrow (x \notin C))\) (T)

3. \((\forall x \in \mathbb{N}) (x \in P) \Rightarrow (x \in C)\)
   
   Let \(P = \{2, 3, 5, 7, \ldots\}\) denote the set of primes. Then:
   
   \((\forall x \in \mathbb{N}) ((x \in P) \Rightarrow (x \in C))\)
   
   is equiv to:
   
   \((\forall x \in \mathbb{N}) (((x \in P) \Rightarrow (x \in C)) \lor ((x \notin C) \Rightarrow (x \notin P)))\)
   
   (F).

4. Consider the following (true) statement: \((\forall x \in \mathbb{R}) [(x > 0) \Rightarrow (\exists y \in \mathbb{R}) (y^2 = x)]\)
   
   We'll put it in negation in positive form:
   
   \(\neg (\forall x \in \mathbb{R}) [(x > 0) \Rightarrow (\exists y \in \mathbb{R}) (y^2 = x)]\)
   
   \((\exists x \in \mathbb{R}) \neg [(x > 0) \Rightarrow (\exists y \in \mathbb{R}) (y^2 = x)]\)
   
   \((\exists x \in \mathbb{R}) [(x > 0) \land \neg (\exists y \in \mathbb{R}) (y^2 = x)]\)
   
   \((\exists x \in \mathbb{R}) [(x > 0) \land \neg (\exists y \in \mathbb{R}) (y^2 = x)]\)
   
   \((\exists x \in \mathbb{R}) [(x > 0) \land ((y^2 = x) \lor \neg (y^2 = x))]\)
\( (\exists x \in \mathbb{R})( (x > 0) \land (\forall y \in \mathbb{R})(y^2 \neq x)) \lor ((x < 0) \land (\exists y \in \mathbb{R})(y^2 = x)) \]

logically equiv. to orig. negated statement (and False)

More useful equivalency

Prop'n: The following equivalences hold:

1. \( p \land (q \lor r) \Leftrightarrow (p \land q) \lor (p \land r) \) (Association Laws)

2. \( p \lor (q \land r) \Leftrightarrow (p \lor q) \land (p \lor r) \) (Distribution Laws)

3. \( p \land (q \lor r) \Leftrightarrow (p \land q) \lor (p \land r) \)

4. \( p \lor (q \land r) \Leftrightarrow (p \lor q) \land (p \lor r) \)

Pf: try the truth tables!

Proving equality of sets using \( \Leftrightarrow \)

There is a strong analogy between the logical connectives and the set operations.

From Ch. 3:

<table>
<thead>
<tr>
<th>Connective</th>
<th>Operation</th>
</tr>
</thead>
<tbody>
<tr>
<td>( p \land q )</td>
<td>( A \cap B )</td>
</tr>
<tr>
<td>( p \lor q )</td>
<td>( A \cup B )</td>
</tr>
<tr>
<td>( p \rightarrow q )</td>
<td>( A \subseteq B )</td>
</tr>
<tr>
<td>( p \Leftrightarrow q )</td>
<td>( A = B )</td>
</tr>
<tr>
<td>( \neg p )</td>
<td>( A )</td>
</tr>
</tbody>
</table>