Chapter 4: Intro to mathematical logic

Goals: - Learn how to write statements more formally (more symbols, fewer words)

- See how the form of a statement suggests the form of its proof.

Recall Def’n (informal) A mathematical statement (or prop’n) is a grammatically correct declarative sentence that is either true or false.

- Consists of words and/or symbols

→ “Statement” can be rigorously defined, but need formal logic.

→ “grammatically correct” also has a precise meaning in that context.

Ex’s

1. Every integer is a real number (T)
2. Every real number is an integer (F)
3. There exists an x ∈ R s.t. x² ≤ 2 (T)
There are infinitely many twin primes (unknown... but either T or F). 

Nonex's ① T! IT (grammatically incorrect/meaningless)
② Shakespeare (not declarative/no truth value)
③ \[x^2 + 1 = 2\]

(a meaningful sequence of symbols asserting an equality... but no truth value unless \(x\) is specified)

called a variable proposition: a sentence that becomes a statement once its variables are specified (or quantified over... more later).

We will use \(P, Q, R, \ldots\) for statements and \(P(x), Q(x, y), \ldots\) for var. props.

E.g. might say: let \(P\) denote "\(5^2 + 1 = 2\)" (F)

Let \(Q(x)\) denote "\(x^2 + 1 = 2\)"

Then \(Q(5)\) is the statement "\(5^2 + 1 = 2\)" (F)
\(Q(1)\) is "\(1^2 + 1 = 2\)" (T)
More var. prop'ns: 1) \( x^2 + 1 \leq 0 \)
2) \( x \in \mathbb{Z} \) and \( x^2 < 39 \)
3) \( z = x + y \)

You should indicate when abbreviating a var. prop'n w/ multiple variables. E.g., could use \( Q(x,y,z) \) to denote 3.

Then: \( Q(1,2,3) \) is "1 = 2 + 3" (F)
\( Q(3,1,2) \) is "3 = 1 + 2" (T)

Quantifiers: other way to turn a var. prop'n into a statement w/ to quantify over its variables.

Ex: "\( x^2 + 1 = 2 \)" is a var. prop'n, but "there exists \( x \in \mathbb{R} \) s.t. \( x^2 + 1 = 2 \)" is a statement (T), or u
"For every \( x \in \mathbb{R} \), we have \( x^2 + 1 = 2 \)" (F)

The clauses "there exists \( x \in \mathbb{R} \) ..." 
"For every \( x \in \mathbb{R} \) ..."
are two types of quantification of the variable \( x \).
- We'll use the symbols

$\forall$ read "for all"

$\exists$ read "there exists"

the "universal quantifier" $\rightarrow$ the "existential quantifier"

- Given a var. prop. $P(x)$ and a set $S$:

"For all $x \in S$, $P(x)$" and

"There exist $x \in S$ such that $P(x)$" are statements

- will denote them by

$\forall x \in S \ P(x)$

$\exists x \in S \ P(x)$

(respectively.

$\exists x \in S \ (x < 5)$

Read "there exist an $x \in N$ s.t. $x < 5"$ (T)
Multiple Quantifiers

5. Consider the var prop'n \( x+y \geq 2 \). Then \((\forall y \in \mathbb{N})(x+y \geq 2)\) is still a var. prop'n but now with only one "free variable," namely \( x \). But \((\forall x \in \mathbb{N})(\exists y \in \mathbb{N})(x+y \geq 2)\) is a (T) statement.

Can also write as \((\forall x, y \in \mathbb{N})(x+y \geq 2)\)
"For all \( x \) and \( y \) in \( \mathbb{N}, x+y \geq 2 \)"

6. Can also mix \( \forall \)'s and \( \exists \)'s, but beware: order of quantifiers is important!

E.g.
\((\forall x \in \mathbb{N})(\exists y \in \mathbb{R})(y^2 = x)\)
"For every \( x \in \mathbb{N} \) there is \( y \in \mathbb{R} \) s.t. \( y^2 = x \)
I.e. every natural number has a real square root (T)

7. \((\forall x \in \mathbb{N})(\exists y \in \mathbb{N})(y^2 = x)\)
I.e. "every natural number has a square root in \( \mathbb{N} \" (F)

What happens if we reverse the order of quantifiers in 6?
get \((\forall y \in \mathbb{R}) (\forall x \in \mathbb{N}) (y^2 = x)\)

i.e. "there is a red monkey s.t. every natural number is equal to \(y^2\)"

- perfectly well-written statement, but absurd and definitely false
- morel: order of quantifiers makes a big deal!

we can also have "inside quantifiers"

- e.g. \((\forall x \in \mathbb{N}) (x > 0 \text{ and } (\exists y \in \mathbb{N}) (y > x))\)
- \((\forall x \in \mathbb{R}) (\text{if } x > 0, \text{ then } (\exists y \in \mathbb{R}) (y^2 = x))\)

are both statements (both CD)

Note: we've insisted all quantified variables range over a specified set

- e.g. \((\forall x \in \mathbb{R})(x^2 \geq 0)\) is meaningful
  - but \((\forall x)(x^2 > 0)\) is not

- what if we want to quantify over variables referring to \(\mathbb{R}\)?

- e.g. to write "For every sets, we have \(\emptyset \subseteq \)"
Symbolically, might try:

\[(\forall x \in \phi (\cdots)) (\phi \subseteq \mathcal{S})\]

set of all sets?  

- but the collection of all sets is not a set (Russell's paradox)
- convention: when quantifying over set variables, we'll write sentence verbally:

1. " For all sets \( \mathcal{S} \) . . . 
   " There exist sets \( \mathcal{A}, \mathcal{B} \) . . . .

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**Connecting + Truth Tables**

- connecting or symbols used to combine multiple statements into one
- all our connecting will be binary (connect two statements into one)
- except negation which is unary
- Truth Tables tell us how truth of a connected statements depends on the truth of the original constituent statements

**Conjunction** ("and") - conjunction of
Two statements P, Q are written

\[ P \land Q \quad \text{"P and Q"} \]

- \( P \land Q \) is true if both P, Q are true

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<thead>
<tr>
<th>P</th>
<th>Q</th>
<th>( P \land Q )</th>
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<tbody>
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<td>T</td>
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Ex's let \( P \) denote:

\[ (\forall x \in \mathbb{Z})(x+1 > x) \]

Let \( Q \) be:

"97 is prime"

Let \( R \) be:

"2+2 = 5."

- then \( P, Q \) are (T) but \( R \) is (F)
- hence \( P \land Q \) is (T)
- but \( P \land R \) and \( Q \land R \) are both (F)
- written out \( P \land Q \) is:

\[ (\forall x \in \mathbb{Z})(x+1 > x) \land (97 \text{ is prime}) \]

Inserting parentheses can clarify an expression.
Disjunction ("or")
- disjunction of statements \( P, Q \)
written \( P \lor Q \) ("\( P \lor Q \)"")
- \( P \lor Q \) is true iff at least one of \( P, Q \) true.

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<tr>
<th>( P )</th>
<th>( Q )</th>
<th>( P \lor Q )</th>
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<tbody>
<tr>
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\( \text{E.g.} \ (\forall x \in \mathbb{R})(x^2 > 0) \lor (96 \text{ is prime}) \)
\( \text{is (T), but} \)
\( (\forall x \in \mathbb{R})(x > 0) \lor (96 \text{ is prime}) \)
\( \text{\( \downarrow \)} \)
\( F \)
\( \lor (F). \)

Negation ("not")
- only unary connective
- negation of a statement \( P \) written \( \neg P \)
- \( \neg P \) true iff \( P \) is false

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<tr>
<th>( P )</th>
<th>( \neg P )</th>
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<tr>
<td>T</td>
<td>F</td>
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<td>F</td>
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\( \text{Ex's} \quad \neg \ (\forall x \in \mathbb{N})(\exists y \in \mathbb{N}) \ (y^2 = x) \)
is (F), hence
(2) \( \neg (\forall x \in \mathbb{N}) (\exists y \in \mathbb{N}) (y^2 = x) \)

is (T), hence:

(3) \( \neg (\forall x \in \mathbb{N}) (\exists y \in \mathbb{N}) (y^2 = x) \)

is (F) again.

(4) For any statement \( P \), the statement

\[ P \lor \neg P \]

is (T)

whereas \( P \land \neg P \) is (F).

- eg. \( (96 \text{ is prime}) \lor \neg (96 \text{ is prime}) \)

is (T)

but \( (96 \text{ is prime}) \land \neg (96 \text{ is prime}) \)

is (F).

- Can use connectives in var. props too.

- eg. Let \( P(x, y) \) denote

\[ " (x > 0) \land (y \text{ is prime}) " \]

then \( P(3,5) \) is (T)

while \( P(3,6) \) is (F)

also \( (\exists x, y \in \mathbb{N}) P(x, y) \) is (T)

but \( (\forall x, y \in \mathbb{N}) P(x, y) \) is (F)