2. Finite sets can be defined by just writing all of their elements in brackets, called roster notation.

- e.g. If \( A = \{2, 4, 6, 11\} \)
  \( B = \{0, *, 11\} \)

Then \( 11 \in A \) and \( 11 \in B \)
while \( 0 \notin B \) but \( 0 \notin A \).

Sets are determined by their elements, order and repetition don't matter.

- e.g. If \( A = \{1, 2, 3, 3\} \)
  then \( A = \{2, 1, 3, 3\} \)
  and \( A = \{1, 1, 2, 3\} \) as well.

3. Sets can be elements of sets!

- e.g. If \( A = \{1, 2, 3\} \) \( B = \{3, 4\} \)
  then \( C = \{A, B\} = \{\{1, 2, 3\}, \{3, 4\}\} \)
  is a legit set.

- Different from \( D = \{1, 2, 3, 4, 3\} \)
  \( (C \) has 2 \( \in\) \( D \) has 4) \).
Some Fundamental Sets

\[ N = \{1, 2, 3, \ldots \} \quad \text{"natural numbers"} \]
(For us, does not include 0)

\[ \mathbb{Z} = \{ \ldots, -2, -1, 0, 1, 2, \ldots \} \quad \text{"integers"} \]

\[ Q = \{ \frac{m}{n} \mid m, n \text{ are in } \mathbb{Z} \text{ and } n \neq 0 \} \quad \text{"rational numbers"} \]

\[ \mathbb{R} = \text{set of real numbers} \]

\[ \mathbb{C} = \text{set of complex numbers} \]

\[ = \{ a + bi \mid a, b \in \mathbb{R} \} \]

So e.g. we have:

- \( 0 \in \mathbb{Z} \) but \( 0 \notin N \)
- \( \frac{22}{7} \in \mathbb{Q} \) but \( \frac{22}{7} \notin \mathbb{Z} \)
- \( \pi \in \mathbb{R} \) but \( \pi \notin \mathbb{Q} \)
- \( i \in \mathbb{C} \) but \( i \notin \mathbb{R} \)

\[ \sqrt{-1} \rightarrow i \]

- the empty set \( \emptyset \) the unique set with no elements
- denoted \( \emptyset \) or \( \{ \} \)
- not the same as \( \{0\} \)

\( \Rightarrow \) this set contains a single element, the empty set contains none.
New Sets from old ones.

Set-builder notation: given a set $X$ and a well-defined property $P$, one can form the set $Y$ consisting of all $x \in X$ with property $P$.

We write $Y = \{ x \in X \mid x \text{ has } P \}$ or $Y = \{ x \in X \mid P(x) \}$ always need to specify the set from which the $x$'s are drawn from.

called "set-builder notation"

Ex's: $\mathbb{E}$ can define $E = \{2, 4, 6, \ldots \}$ by:

$E = \{ n \in \mathbb{N} \mid n \text{ is a multiple of } 2 \}$

or, more symbolically:

$E = \{ n \in \mathbb{N} \mid \text{there is } k \in \mathbb{N} \text{ s.t. } n = 2k \}$

$\mathbb{O}$ Once $E$ is defined, can use it to define other sets:

$E \cup \mathbb{O} = \{ n \in \mathbb{N} \mid \text{there is } k \in \mathbb{E} \}$

$s.t. \ n = k - 13$

$= \{1, 3, 5, 7, \ldots \}$
③ the set over which you range is important.
\[ \{ x \in \mathbb{R} \mid x^2 - 2 = 0 \} = \{ \sqrt{2}, -\sqrt{2} \} \]

whereas
\[ \{ x \in \mathbb{Z} \mid x^2 - 2 = 0 \} = \emptyset \]

since no integers satisfy \( x^2 - 2 = 0 \).

More notation: for a given \( n \in \mathbb{N} \),
\[ \{ n \} \]

denotes the set \( \{ 1, 2, \ldots, n \} \)
- e.g. \( \{ 5 \} = \{ 1, 2, 3, 4, 5 \} \)

Subsets:
- a set \( Y \) is a subset of \( X \) if for every \( y \in Y \) we have \( y \in X \).
- in this case we write \( Y \subseteq X \).
- \( Y \) is a proper (or strict) subset of \( X \) if \( Y \subseteq X \) but \( Y \neq X \).
- we (sometimes) write
  \[ Y \subset X \text{ or } Y \subset X \]
  to indicate "\( Y \) is a proper subset of \( X \)."
whereas \( Y \nsubseteq X \) means "\( Y \) is not a subset of \( X \)"

\[
\text{Ex}5 \quad \begin{align*}
&\text{(1) } \{1,3,3\} \subseteq \{1,2,3,4,3\} \\
&\text{why: } 1 \in \{1,2,3,4,3\} \text{ and } 3 \in \{1,2,3,4,3\} \\
&\text{so it is a proper subset, so we could write } \{1,3,3\} \subset \{1,2,3,4,3\} \\
&\text{or } \{1,3,3\} \subset \{1,2,3,4,3\}
\end{align*}
\]

\[
\text{② } \{1,3,3\} \nsubseteq \{1,2,3,4,3\} \\
\text{why: } -3 \notin \{-5,3,3\} \text{ but } -5 \notin \{1,2,3,4,3\}
\]

\[
\text{③ } \mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}.
\]

Notice: "\( \subseteq \)" is a \underline{transitive} relation, i.e. if \( A \subseteq B \) and \( B \subseteq C \) then \( A \subseteq C \).

Let's prove this using the def'n of \( \subseteq \).
Prop. 1: For any sets \( A, B, C \), if \( A \subseteq B \) and \( B \subseteq C \) then \( A \subseteq C \).

\textbf{Pf.} - Suppose \( x \in A \) is an arbitrary element of \( A \).
- Since \( A \subseteq B \), we have \( x \in B \), by definition of \( \subseteq \).
- Then, since \( B \subseteq C \), we have \( x \in C \), too, again by definition of \( \subseteq \).
- Since \( x \in A \) was arbitrary, the same argument would apply to any element of \( A \).
- Hence, every element of \( A \) is also an element of \( C \), i.e., \( A \subseteq C \).

More ex's:
① For any set \( X \), we have \( X \subseteq X \). Pf. Fix \( x \in X \). Then \( x \in X \) too...
② Set-builder notation defines a subset, i.e., if \( Y = \{ x \in X | x \text{ has property P} \} \) then \( Y \subseteq X \).
③ For any set \( X \) we have \( \emptyset \subseteq X \), perhaps unintuitive, but here's why:
Pr: It is true that
if \( i \) \( x \in \emptyset \)
then \( (i) \) \( x \in X \)

Simply because \( (i) \) never holds!

**Operations on Sets**

**Intersections:** the intersection of two sets \( A, B \), denoted \( A \cap B \), is the set of elements belonging to both \( A \) and \( B \),

i.e. \( x \in A \cap B \) if (and only if)
\( x \in A \) and \( x \in B \).

![Venn diagram](image)

**EX:** if \( A = \{1, 2, 3, 4, 5\} \) then: \( A \cap B = \{1, 3\} \)
\( A \cap C = \{2, 4, 3\} \)
\( B \cap C = \emptyset \).

Def: Two sets are called **disjoint** if
their intersection is \( \emptyset \).

ex: \( B, C \) above are disjoint
\( B \cap C = \emptyset \).
Prop' n: For any sets $A, B$ we have:

(i) $A \cap B \subseteq A$
(ii) $A \cap B \subseteq B$

Is "obvious" from the picture, but let's practice proving from the def'n.

PF: (i) Fix $x \in A \cap B$
    - then $x \in A$ and $x \in B$, by def'n of $\cap$,
    - hence in particular $x \in A$.
    - Since $x \in A$ was arbitrary,
    - every el't of $A \cap B$ is an el't of $A$,
    - i.e. $A \cap B \subseteq A$.

(ii) Similar.

Unions

Def'n: the union of $A$ and $B$, denoted $A \cup B$, is the set of el'ts contained in either $A$ or $B$,
    i.e. $x \in A \cup B$
iff $x \in A$ or $x \in B$

Note: "or" here (as in all math) is non-exclusive.
Ex5 ① \( \{1,3,5,7\} \cup \{2,4,6\} = \{1,2,3,4,5,6,7\} = \mathbb{N} \).

② If \( O = \{1,3,5,\ldots,3\} \) and \( E = \{2,4,6,\ldots,3\} \),
then \( O \cup E = N = \{1,2,3,4,\ldots,3\} \).

③ Prop'n: For any sets \( A, B \),
we have: (i) \( A \subseteq A \cup B \)
(ii) \( B \subseteq A \cup B \)

PF: you try.

**Difference**

Def'n the difference of two sets \( A \) and \( B \), denoted \( A - B \), is the set of elements in \( A \) that are not in \( B \).

i.e. \( x \in A - B \)
if \( x \in A \) and \( x \notin B \).
Ex: if $A = \{1, 2, 3, 3\}$
\[B = \{3, 4, 5, 3\}\]

then $A - B = \{1, 2\}$  $B - A = \{4, 5\}$

Notice: difference is not a commutative operation, i.e. $A - B \neq B - A$ in general.

However, $\cap$ and $\cup$ are commutative, i.e. we always have $A \cup B = B \cup A$  $A \cap B = B \cap A$.

Note: in defining $\cap, \cup$, it is sometimes convenient to assume our sets $A, B$ are both subsets of a larger set $U$ called a universal set.
then we can define these operations using set-builder notation.

\[ A \cap B = \{ x \in U \mid x \in A \text{ and } x \in B \} \]
\[ A \cup B = \{ x \in U \mid x \in A \text{ or } x \in B \} \]
\[ A - B = \{ x \in U \mid x \in A \text{ and } x \notin B \} \]
\[ = \{ x \in A \mid x \notin B \} \]

**Complement**

Def'n given a set \( A \), and a universal set \( U \) with \( A \subseteq U \), the complement of \( A \), denoted \( \overline{A} \), is the set of elements in \( U \) that are not in \( A \).

\[ \overline{A} = \{ x \in U \mid x \notin A \} \]

Ex: 1) \( S = \mathbb{N} \)
\[ A = \{ 1, 2, 3 \} \]
\[ E = \{ 2, 4, 6, \ldots \} \]
\[ O = \{ 1, 3, 5, \ldots \} \]

Note: really \( \overline{A} \) is just \( U - A \).