Functions

Functions, like relations, ubiquitously appear in math. But what are functions? An intuitive definition: a rule that assigns to each $x$ in a domain $A$ a unique output $f(x)$ in codomain $B$.

We can define them rigorously as a special type of relation.

**Definition**: A function (with domain $A$ and codomain $B$) is a relation $f \subseteq A \times B$ such that

- For every $a \in A$, there is a unique $b \in B$ s.t. $(a, b) \in f$.

Can write:

$$\forall a \in A \left( \exists! b \in B \cdot (a, b) \in f \right) \land$$

$$\left( \forall c \in B \right) \left[ (a, c) \in f \Rightarrow c = b \right]$$

We write $f : A \rightarrow B$ to indicate $f \subseteq A \times B$ is a function.
- will use standard notation \( f(c) = b \) to mean \((c, b) \in f\).

Note: definition says every \( c \in A \) is assigned an output \( f(c) \in B \).
- does not insist for every \( b \in B \) there is \( c \in A \) s.t. \( f(c) = b \) (function by this property are called onto or surjective).

**Ex:**

1. Let \( A = \{1, 2, 3\} \)
\[
B = \{*, \emptyset, 0, 3\}
\]

Then \( f = \{(1, *), (2, \emptyset), (3, 3)\} \) is a function from \( A \) to \( B \).

![Diagram]

But \( g = \{(1, *), (1, \emptyset), (2, *), (3, 3)\} \) is not (1 does not have a unique \( g(c) \in B \)).
For \( u, v, \in \mathbb{Z} \), \((3, \varnothing) \in 3\), since 1 is not assigned an output.

2. We'll often define functions by some rule, as a common practice. E.g.

\[ f: \mathbb{R} \rightarrow \mathbb{R} \]
\[ f(x) = x^2 \]

or

\[ g: \mathbb{R} \rightarrow \mathbb{Z} \]
\[ g(x) = \lfloor x \rfloor \]

but behind the scenes, still consider \( f, g \) to be sets.
of ordered pairs (e.g. \((2, 4) \in f\), \((11, 3) \in g\))

One issue that arises when defining functions by rules is that sometimes such rules do not yield well-defined functions.

E.g. Suppose we define \(f: \mathbb{Q} \to \mathbb{Z}\) by the rule \(f(\frac{m}{n}) = m\text{th}\).

Then this "function" is not one:

\[ f(\frac{1}{2}) = 1 + 2 = 3 \neq 6 = 2 + 4 = f(\frac{2}{4}) \]

But \(\frac{1}{2} = \frac{2}{4}\).

So \(f\) assigns multiple outputs to some input.

What's really going on here is that there's an implicit equivalence relation on fractions (\(\frac{1}{2} = \frac{3}{6} = \frac{6}{-12}\)) but we're defining \(f\) on the \(\text{representatives of the}\)
-equivalent classes.

- In general, when given a rule defining some relation \( f: A \times B \to \mathbb{R} \) to check if it is a function one must verify:
  1. That \( A \) and \( B \times B \) s.t. \( (a, b) \in f \)
  2. If \( a = a' \) then \( f(a) = f(a') \).

Equality of functions

Q: What does it mean for functions \( f: A \to B \) and \( g: A \to B \) to be equal?

Well, we've defined \( f, g \) as sets of ordered pairs, i.e., \( f, g \subseteq A \times B \), so they're equal if they're equal as sets.

I.e.,

\[ f = g \quad \text{iff} \quad (f \subseteq g \text{ and } g \subseteq f) \]

I.e.,

\[ (a, b) \in f \quad \text{iff} \quad (a, b) \in g \]
However, in practice it is easier to use the following criterion:

**Theorem** If $f : A \to B$ and $g : A \to B$ are functions, then $f = g$ if and only if for every $a \in A$, $f(a) = g(a)$.

**PF.** Exercise

**Main Point:** Functions may be equal despite being defined by different rules.

**Ex.** Let $A = \{1, 2, 3\}$ define $f : A \to \mathbb{N}$ and $g : A \to \mathbb{N}$ by $f(x) = x^2 + 11x$, $g(x) = 6x^2 + 6$.

Then $f(1) = 12$, $g(1) = 12$, $f(2) = 30$, $g(2) = 30$.

Then $f(3) = 60$, $g(3) = 60$.

Hence $f = g$!
Images

Though a function $f$ need not "hit" every value in its codomain, we do have a name for the set of outputs:

**Def'n**: Suppose $f : A \rightarrow B$ is a function and $x \in A$.

The **image of $x$ under $f$** is defined as:

$$\text{Im}_f(x) = \{ b \in B \mid (\exists a \in A) \ f(a) = b \}$$

or more informally

$$\text{Im}_f(x) = \{ f(a) \mid a \in x \}$$

When $X = A$, we say simply that $\text{Im}_f(A)$ is the **image of $f$** and sometimes just write $\text{Im}_f$.  

**Def'n** says: $\text{Im}_f(x)$ is the **set of outputs of $f$ at point $x$**:  

$$\text{Im}_f = \text{Im}_f(A) = \text{set of all output values at point } x$$
Let $A = \{1, 2, 3\}$ and $B = \{\ast, 0, \Delta\}$ with $f = \{(1, \ast), (2, 0), (3, \ast)\}$.

Then: $\text{Im}_f(\{1, 3\}) = \{f(1), f(3)\}$
    $\quad\quad\quad = \{\ast, \ast\}$
    $\quad\quad\quad = \{\ast\}$
$\text{Im}_f = \text{Im}_f(A) = \{f(1), f(2), f(3)\}$
    $\quad\quad\quad = \{\ast, 0, \ast\}$

Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be defined by $f(x) = x^2$.

Then: $\text{Im}_f(\{-1, 0, 1\}) = \{0, 1\}$
$\text{Im}_f = \{x \in \mathbb{R} \mid x \geq 0\}$
- Functions add a layer of complexity to the basic set theory of \( \land, \lor, \) etc. we studied earlier.

- e.g.

Prop.: Suppose \( f : A \to B \) is a function and \( S, T \subseteq A \).

Then:

\[ \emptyset \subseteq \text{Im}f(S \cap T) \subseteq \text{Im}f(S) \cap \text{Im}f(T) \]

PF: Fix \( y \in \text{Im}f(S \cap T) \)

- then \( \exists x \in S \cap T \) s.t. \( f(x) = y \)
- hence \( x \in S \) and \( x \in T \)
- hence \( f(x) \in \text{Im}f(S) \) and \( f(x) \in \text{Im}f(T) \)

i.e. \( y \in \text{Im}f(S) \) and \( y \in \text{Im}f(T) \)

i.e. \( y \in \text{Im}f(S) \cap \text{Im}f(T) \)

Since \( y \) was arbitrary the theorem is proved.
In general the container cannot be reversed:
- e.g. \( \text{Consider } f(x) = x^2 \text{ on } \mathbb{R} \).
  - Let \( S = \{ -1, 0 \} \)
  - \( T = \{ 0, 1, 2 \} \)
    - So \( SAT = \{ 0 \} \)

Then: \( \text{Im}_f(S) = \{ f(-1), f(0) \} = \{ 1, 0 \} \)
\( \text{Im}_f(T) = \{ f(0), f(1), f(2) \} = \{ 0, 1, 4 \} \)

\( \text{So } \text{Im}_f(S) \cap \text{Im}_f(T) = \{ 0, 1 \} \)

But \( \text{Im}_f(SAT) = \text{Im}_f(\{ 0 \}) = \{ f(0) \} = \{ 0 \} \)

So \( \text{Im}_f(SAT) \neq \text{Im}_f(S) \cap \text{Im}_f(T) \)

Fact that we don’t have equality in above property is an essential feature of functions; multiple inputs can have the same output.
Define Suppose $f: A \rightarrow B$ is a function and $Y \subseteq B$. Then the preimage of $Y$ under $f$ is defined as:

\[ \text{Pre} \text{Im} f(Y) = \{ x \in A \mid f(x) \in Y \} \]

Note: Since $f(x) \in B$ for every $x \in A$, we don't separately define $\text{Pre} \text{Im} f(\emptyset)$, since this always $= \emptyset$. 
ex: ① \( A = \{1, 2, 3\} \)
\( B = \{0, 1, 0\} \)
\( f = \{(1, \ast), (2, \ast), (3, \ast)\} \)

\[ \text{then: } \text{PreIm}(\{\ast\}) \]
\[ = \{x \in A \mid f(x) \in \{\ast\}\} \]
\[ = \{x \in A \mid f(x) = \ast\} \]
\[ = \{1, 3\} \]
\[ \text{PreIm}(\{\ast, 0\}) \]
\[ = \{x \in A \mid f(x) \in \{\ast, 0\}\} \]
\[ = \{1, 2, 3\} \]
\[ \text{PreIm}(\{\{\}\}) \]
\[ = \emptyset \]

Note: when computing \( \text{PreIm}(Y) \) read it as that every \( y \in Y \) \( u \) in image of \( A \)!

② let \( f : \mathbb{R} \to \mathbb{R} \) be \( f(x) = x^2 \).

\[ \text{then: } \text{PreIm}(\{10, 1\}) \]
\[ = \{x \in \mathbb{R} \mid f(x) \in \{10, 1\}\} \]
\[ = \{x \in \mathbb{R} \mid x^2 \in \{10, 1\}\} \]
\[ = \{\pm 1, 0\} \]
\[
\text{Pre} \text{ Inf} (C_0, 2) \\
= \{ x \in \mathbb{R} \mid x^2 \in [0, 2] \} \\
= \{ x \in \mathbb{R} \mid 0 \leq x^2 \leq 2 \} \\
= \{ x \in \mathbb{R} \mid -\sqrt{2} \leq x \leq \sqrt{2} \} \\
= C - [\sqrt{2}, \sqrt{2}] \\
\text{Pre} \text{ Inf} (C_0, 0) \\
= \{ x \in \mathbb{R} \mid x^2 \in (0, \infty) \} \\
= \{ x \in \mathbb{R} \mid x^2 > 0 \} \\
= \mathbb{R} \\
\text{Pre} \text{ Inf} (\mathbb{R}) = \mathbb{R}.
\]

Can play with Images and Preimages.

Prop'n Suppose \( f : A \to B \) is a function.

(i) Fix \( X \subseteq A \), then \( \text{Pre} \text{ Inf} (\text{Inf}(x)) = X \)

(ii) Fix \( Y \subseteq B \), then \( \text{Inf} (\text{Pre} \text{ Inf}(y)) \subseteq Y \)
PF: (i) \(\text{Fix } x \in X\)
- \(\forall y = f(x)\)
- \(\text{So } y \in \text{Im } f(X)\)
- \(\exists z \in \text{Im } f(X) \text{ s.t. } f(z) = x\)
  
  namely \(z = y\)
- \(\text{hence } x \in \text{PreIm } (\text{Im } f(x))\)

\(\therefore \text{Since } x \text{ was arbitrary,}\)
\(\text{statement is proved.}\)

(ii) \(\text{Fix } y \in \text{Im } (\text{PreIm } (Y))\)
- \(\exists z \in \text{PreIm } (Y) \text{ s.t. } f(z) = y\)
  
  \(\text{but } f(z) \in \text{Im } f(X)\)
- \(\therefore y \in \text{Im } f(X)\)

\(\therefore \text{Since } y \text{ was arbitrary,}\)
\(\text{statement is proved.}\)

\[\text{Picture:}\]

(i) \(\text{PreIm } (\text{Im } f(x))\)

- \(X \to \text{Im } f(x)\)
- \(A \to B\)
Neither containment can be reversed.

E.g., \(-\mathbb{W} \xrightarrow{f} \mathbb{R} \rightarrow \mathbb{R}\) by \(f(x) = x^2\)
\[-\mathbb{W} \xrightarrow{f} \mathbb{R} \rightarrow \mathbb{R}\]
\[X = \{-1, 1\}\]

Thus: \(\text{Imp}(X) = \text{Imp}(\{1\}) = \{f(1)\} = \{1\}\)

\(\text{PreImp}(\text{Imp}(X)) = \text{PreImp}(\{1\}) = \{-1, 1\}\)
So: \( x \notin \text{PreIm}_f(\text{Im}_f(x)) \)

Now: \( \forall y \in \{ -5, 1 \} \)

- Then \( \text{PreIm}_f(y) \)

\[
\begin{align*}
\text{PreIm}_f(\{ -5, 1 \}) &= \{ x \in \mathbb{R} \mid f(x) \in \{ -5, 1 \} \} \\
&= \{ x \in \mathbb{R} \mid x^2 \in \{ -5, 1 \} \} \\
&= \{ -1, 1 \}
\end{align*}
\]

So: \( \text{Im}_f(\text{PreIm}_f(y)) \neq \emptyset \)

\[
\begin{align*}
\text{Im}_f(\{ -1, 1 \}) &= \{ 1 \} \\
\text{Im}_f(\text{PreIm}_f(y)) &\neq y.
\end{align*}
\]
Injections

\[ \begin{align*}
A &= \{1, 2, 3\} \\
B &= \{*, \heartsuit, \odot\} \\
C &= \{1, 2\} \\
D &= \{*, \heartsuit, \odot, \Delta\}
\end{align*} \]

define \( g : A \rightarrow B \) \nonumber

\( h : C \rightarrow D \) \nonumber

\( j : A \rightarrow D \) \nonumber

by: \( g = \{(1, *), (2, \heartsuit), (3, *)\} \)
\( h = \{(1, \heartsuit), (2, \Delta)\} \)
\( j = \{(1, *), (2, \heartsuit), (3, \Delta)\} \)
Surjective

A function \( f: A \rightarrow B \) is \textit{surjective} (or \textit{onto}) if \( \text{Im} f = B \)

that is, iff

\[
(\forall b \in B)(\exists a \in A) \ f(a) = b
\]

ex: \(-g, j\) above are \textit{surjective}

- \( h \) \( \notin \) \( \text{Im} \)

Proving \textit{surjectivity}

ex: 1. Define \( f: \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Z} \)

by \( f((m,n)) = m+n \)

Claim \( f \) \textit{surjective}

\[ \text{PF: } \text{WTS} \ (\forall x \in \mathbb{Z})(\exists (m,n) \in \mathbb{Z} \times \mathbb{Z}) \]

\( s.t. \ f(m,n) = x \)

- So fix \( x \in \mathbb{Z} \)

- observe \( f(0,x) = 0 + x = x \)

- hence \( \exists (m,n) \) s.t. \( f(m,n) = x \)

namely \( (m,n) = (0,x) \)
- Since $x$ was arbitrary, the theorem is proved.

2. **Define** $f : \mathbb{R} \rightarrow \mathbb{R}$ by
   \[ f(x) = 2x + 1 \]

   **Claim**: $f$ is surjective

   **PF**: Fix $y \in \mathbb{R}$

   \[-\text{let } x = \frac{y-1}{2} \]

   - then
     \[ f(x) = f\left(\frac{y-1}{2}\right) \]
     \[ = 2\left(\frac{y-1}{2}\right) + 1 \]
     \[ = y \]

   - since $y$ was arbitrary,
     the claim is proved

3. **Define** $f : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$ by
   \[ f(m, n) = m + n \]

   **Claim**: $f$ is not surjective
**Proof:** WTS: 
\[(\exists x \in \mathbb{N}) (\forall (m,n) \in \mathbb{N} \times \mathbb{N}) f(mn) \neq x\]

- Consider \(x = 1\)
- Fix \((m,n) \in \mathbb{N}\n\)
- Then since \(m,n \geq 1\)
  \(\Rightarrow\) we have \(m+n \geq 2\)
- That is, \(f(mn) \geq 2\)
- Hence \(f(mn) \neq 1\)
- Since \((m,n)\) was arbitrary
  the claim is proved.

---

**Injective**

**Definition:** Suppose \(f: A \rightarrow B\)

... a function.

Then \(f\) is called injective

(or 1-1, or one-to-one) if

\[(\forall x, y \in A) (f(x) = f(y) \Rightarrow x = y)\]

---

"Sometimes helpful to write this definition in contrapositive form"

\[(\forall x, y \in A) (x \neq y \Rightarrow f(x) \neq f(y))\]

"distinct inputs map to distinct outputs"
ex's - $g$ above is not injective:
1) $h \neq j \Rightarrow g(h) = g(j) = *$
- $h, j$ are injective.

Proving injectivity

ex. 7) Define $f: \mathbb{R} \rightarrow \mathbb{R}$ by
$f(x) = 3x + 6$.
Claim $f$ is injective.

Two approaches:
Fix $x, y \in A$ and either
1) Assume $f(x) = f(y)$, prove $x = y$
2) Assume $x \neq y$, prove $f(x) \neq f(y)$

ex'ly 7) Define $f: \mathbb{R} \rightarrow \mathbb{R}$ by
$f(x) = 3x + 6$.

Claim $f$ is injective.

Pf. - Fix $x, y \in \mathbb{R}$.
- Assume $f(x) = f(y)$
- Then $3x + 6 = 3y + 6$
  $\Rightarrow 3x = 3y$
  $\Rightarrow x = y$
- Since \( x, y \) were arbitrary, the claim is proved.

2. Define \( f : \mathbb{N} \to \mathbb{N} \) by \( f(n) = n^2 \)

Claim \( f \) is injective.

\[ \text{PF:} \quad \text{Fix } n, m \in \mathbb{N} \]
- Assume \( n \neq m \).
- Then either \( n < m \) or \( m < n \).
- Assume, without loss of generality, that \( n < m \).
- Then since \( n, m \in \mathbb{N} \) are both positive, we may square both sides to obtain \( n^2 < m^2 \).
- Hence \( f(n) < f(m) \).
- Hence \( f(n) \neq f(m) \). \checkmark
- Claim is proved.

3. Define \( f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \) by \( f(mn) = m + n \)

Claim \( f \) is not injective.
PF: - Observe $F(1, 3) = 4 = F(2, 2)$
- but $(1, 3) \neq (2, 2)$
- hence $F$ is not injective

Bijections

Define a function $F: A \rightarrow B$ is called bijective if $F$ is both injective and surjective.

Ex: g above is not a bijection (is a surjection but not injective)
- h isn't either (is an injection but not a surjection)
- j is a bijection

Proving bijection

Ex: Define $F: IR \rightarrow IR$ by $F(x) = 3x - 1$

Claim $F$ is a bijection
PF: Subclaim: $F$ is an injection
PF: Fix $x, y \in IR$ and assume $F(x) = F(y)$
then \( 3x - 1 = 3y - 1 \)
\( \Rightarrow \) \( 3x = 3y \)
\( \Rightarrow \) \( x = y \)
\( \sqrt{ } \)
Subclaim proved.

Subclaim: \( f \) is a surjection.

Pf.: Fix \( y \in \mathbb{R} \)

\( \Rightarrow x = \frac{y + 1}{3} \)

Thus:
\( f(x) = f(\frac{y + 1}{3}) \)
\( = 3(\frac{y + 1}{3}) - 1 \)
\( = y \)

- Since \( y \in \mathbb{R} \) was arbitrary,

subclaim is proved.

Hence \( f \) is a bijection, as claimed.

(2) Define a function \( f: \mathbb{Z} \to \mathbb{N} \)

say,
\[ f(n) = \begin{cases} 2^n & \text{if } n > 0 \\ 2(n) + 1 & \text{if } n \leq 0 \end{cases} \]
Claim $F$ is a bijection

PF: **Surjectivity**
- Fix $n \in \mathbb{N}$
- If $n$ is even, then $n = 2k$ for some $k \in \mathbb{N}$ (so $k > 0$)
  - hence $F(k) = 2k = n$
- If $n$ is odd, then $n = 2k + 1$ for some $k \in \mathbb{N} \cup \{0\}$
  - hence $-k \leq 0$
  - hence $F(-k) = 2k + 1 = n$
- hence in either case $F(x) = n$
  - hence $F$ is surjective
(iii) injectivity

- Fix $n, m \in \mathbb{Z}$ and suppose $n \neq m$
- WLOG assume $n < m$

**Case 1:** $0 < n < m$
- then $f(n) = 2n$ and $f(m) = 2m$
- since $n < m$, $2n < 2m$
- i.e. $f(n) < f(m)$
- hence $f(n) \neq f(m)$

**Case 2:** $n \leq 0 \leq m$
- then $f(n) = 2(-n) + 1$
- $f(m) = 2(-m) + 1$
- since $n < m$ we have
  - $-n > -m$
  - $\Rightarrow 2(-n) > 2(-m)$
  - $\Rightarrow 2(-n) + 1 > 2(-m) + 1$
  - i.e. $f(n) > f(m)$
  - hence $f(n) \neq f(m)$

**Case 3:** $n \leq 0 < m$
- then $f(n)$ is odd
- hence $f(n) \neq f(m)$
Thus in all 3 cases

\[ f(n) \neq f(m) \]

- hence \( f \) is injective.
- hence \( f \) is bijective.

**Compositions + Inverses**

**Defn** Suppose \( P : A \to B \) and \( q : B \to C \) are functions. The composition \( g \circ f \) of \( f \) and \( g \) denoted \( g \circ f \) is the function defined by,

\[ g \circ f(x) = g(f(x)) \]

\[ \text{Diagram:} \]

A \[\xrightarrow{f} B \xrightarrow{g} C\]
\[ \text{Def'ln} \ F: \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \text{ by} \]
\[ F(m, n) = mn \]
\[ \text{Def'ln} \ g: \mathbb{Z} \to \mathbb{N} \text{ by} \]
\[ g(n) = n^2 + 1 \]

Then: \[ g \circ F : \mathbb{Z} \times \mathbb{Z} \to \mathbb{N} \]
\[ g \circ F (2, 1) = g(F(2, 1)) \]
\[ = g(2) \]
\[ = 2^2 + 1 = 10 \]

In general
\[ g \circ F (m, n) = g(F(m, n)) \]
\[ = g(mn) \]
\[ = (mn)^2 + 1 \]

The identity

\[ \text{Def'ln} \ \text{Let } A \text{ be a set, the identity function on } A \text{, denoted} \]
\[ \text{id}_A \text{, is the (unique) function on } A \text{ such that} \]
\[ \text{id}_A (x) = x \]
\[ \text{"} F(x) = x \text{"} \]

\[ \text{Note: } \text{id}_A : A \to A. \]
**Inverses Def'n.** Let \( f: A \to B \) be a function. Then \( f \) is invertible if and only if there exists a \( g: B \to A \) s.t.

\[
\begin{align*}
g \circ f &= \text{id}_A \\
f \circ g &= \text{id}_B 
\end{align*}
\]

- \( g \) is called the inverse of \( f \)
- and written \( f^{-1} \).

**Note:** Not all functions are invertible!

**In fact we have:**

**Theorem** Let \( f: A \to B \) be a function. Then \( f \) is invertible if and only if \( f \) is a bijection.

**Proof:** \((\Rightarrow)\) Suppose \( f \) is invertible. Let \( g \) be \( f \)'s inverse. We prove \( f \) is a bijection.

**Surjectivity:** For \( y \in B \), consider \( x = g(y) \). Then \( f(x) = f(g(y)) = g(y) = \text{id}_B(y) = y \).

So \( \exists x \text{ s.t. } f(x) = y \) (namely \( y = g(y) \)).
Injectivity: Suppose \( x, y \in A \) and \( f(x) = f(y) \).
- Then \( g(f(x)) = g(f(y)) \).
- But since \( gof = id \) this means \( x = y \).

\((\Leftarrow)\): Suppose \( f \) is a bijection from \( A \) to \( B \).
- Define \( g = \{ (b, a) \in B \times A \mid (a, b) \in f \} \).

We prove \( g = f^{-1} \).

Claim 1: \( g \) is a function from \( B \) to \( A \).
- Fix \( b \in B \). Since \( f \) is surjective, there is a unique \( a \in A \) s.t. \((b, a) \in f \).
- \( g\) is defined at \( b \) since \( (b, a) \in f \).

Claim 2: \( g \) is the inverse of \( f \).
- Suppose \( a \in A \) s.t. \((b, a) \in f \).
  - \( g(a) = \{ (b, a), (c, a) \} \) implies \( (a, b) \in f \).
  - \( f(a) = b \).

Since \( F \) is injective \( \text{w.h.p} \) be \( a = \alpha \)

\text{Claim 2: } g = F^{-1}.

\text{pf: } \text{Fix } a \in A, \text{ let } b = F(a)
\text{then } (a, b) \in F
\text{hence } (b, a) \in g \text{ i.e. } g(b) = a
\text{then } g(F(a)) = a \text{ so } g \circ F = \text{id}_A
\text{Fix } b \in B, \text{ let } a = g(b)
\text{so } (b, a) \in g. \text{ Then } (a, b) \in F
\text{i.e. } F(a) = b.
\text{so } F \circ g = \text{id}_B.

Can use theorem to prove Certain functions are bijection.

\text{Ex: 1 Define } F: \mathbb{R} \rightarrow \mathbb{R}
\text{by } F(x) = 2x - 5
\text{claim } F \text{ is a bijection}
\text{pf: } \text{we'll show } F \text{ has an inverse}
- \text{let } g(x) = \frac{x + 5}{2}
\text{Fix } x \in \mathbb{R}
\text{then } F \circ g(x) = F(g(x)) = 2(\frac{x + 5}{2}) - 5
= x
\[ g \circ f(x) = g(f(x)) = \frac{(2x-5)+5}{2} = x \]

Here \( g \circ f = f \circ g = \text{id}_{\mathbb{R}} \) when \( g \circ f \) is invertible.
Here \( f \circ g \) is a bijection.