Homework #7

1. (Constructing the rationals) Define a relation $\sim$ on $\mathbb{Z} \times \mathbb{N}$ such that for any two pairs $(a, b), (c, d) \in \mathbb{Z} \times \mathbb{N}$ we have:

$$(a, b) \sim (c, d) \iff ad = bc$$

a. Prove that $\sim$ is an equivalence relation

b. Determine the set $[(0,3)]_\sim$. That is, define a set using set-builder notation and then prove that this set is $[(0,3)]_\sim$.

c. Write out three elements of $[(2,5)]_\sim$.

d. We can naturally identify $(\mathbb{Z} \times \mathbb{N})/\sim$ with one of our standard sets. Which set is this?

2. For a fixed $r \in \mathbb{R}$, the graph of the equation $y = r$ is a horizontal line in the plane $\mathbb{R}^2$. The collection of all such lines is a partition of $\mathbb{R}^2$. Define an equivalence relation on $\mathbb{R}^2$ whose equivalence classes are precisely these horizontal lines.

3. (Modular arithmetic) A key property of the relation of congruence modulo $n$ is that it is preserved by addition and multiplication. In this sense, congruence behaves like equality. For example, from the relation $2 \equiv 5 \pmod{3}$ we can, by adding 13 to both sides, deduce $15 \equiv 18 \pmod{3}$. And by multiplying both sides by 2 we obtain $4 \equiv 10 \pmod{3}$.

Prove that this works in general. That is, fix $n \in \mathbb{N}$ and prove that for any $x, y, k \in \mathbb{Z}$ we have

i. if $x \equiv y \pmod{n}$, then $k + x \equiv k + y \pmod{n}$

ii. if $x \equiv y \pmod{n}$, then $kx \equiv ky \pmod{n}$

4. Let $A$ be a set and suppose $R$ is a partial order on $A$ (that is, $R$ is a reflexive, transitive, and antisymmetric relation on $A$). For $x \in A$ define the cone of $x$, denoted $(x)_R$, as follows

$$(x)_R = \{a \in A \mid (a, x) \in R\}$$

Prove that for all $x, y \in A$, we have $(x)_R \subseteq (y)_R$ if and only if $(x, y) \in R$.

5. Let $A$ be a set and suppose $R$ is an equivalence relation on $A$. Prove that set of equivalences classes, $A/R$, is a partition of $A$. 