Notice: in both examples 1 and 2, the set of equiv. classes forms a partition of the underlying set \( A \) (in 1, \( R \) in 2).

It turns out this is always the case:

**Theorem:** If \( R \) is an equiv. relation on \( A \), then \( A/R \) is a partition of \( A \).

**PF:** HW. For a hint, see 6.7.13 on pg. 449, which outlines an approach to the proof.

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**Partitions yield Equiv. relations**

**Idea:** if \( P \) is a partition on \( A \), we can define an equiv. relation \( R \) on \( A \) by rule “\( (x,y) \in R \) iff \( x \) and \( y \) are in some piece of partition.”

**Picture:**

\[ (x,y) \in R \]

but \( (x,z) \notin R \).
Let's prove this work:

**Theorem**  $S_P \subseteq P$ is a partition of $A$.

Define a relation $R_{PP}$ on $A$ by:

$$(x, y) \in R_{PP} \text{ if } \exists x \in P \text{ such that } x \in x \text{ and } y \in x.$$

Then, $R_{PP}$ is an equivalence relation.

**PF:** (i) (Reflexivity)
- Fix $x \in A$.
- Since $P$ is a partition we know $U_x = A$.
- So since $x \in A = \bigcup_{x \in P} x$.
- Hence $x \in x$.
- Hence $(x, x) \in R_{PP}$.

(iii) (Symmetry)
- Fix $x, y \in A$ and suppose $(x, y) \in R_{PP}$.
- By def'n of $R_{PP}$, there is some $x \in P$ such that $x \in x$ and $y \in x$.
- Hence $y \in x$ and $x \in x$.
- Hence $(y, x) \in R_{PP}$.
(iii) Transitivity

- Fix $x, y, z \in A$ and suppose $(x, y) \in R_p$ and $(y, z) \in R_p$.
- Then by def'n of $R_p$, there is some $x \in P$ s.t. $x \in x$ and $y \in x$.
- Also, there is some $y \in P$ s.t. $y \in Y$ and $z \in Y$.
- Hence $y \in x \land y$.
- In particular, $x \land y \neq \emptyset$.
- But then $X = Y$, since $P$ is a partition.
- Hence $x \in x \land z \in x$.
- Hence $(x, z) \in R_p$.

Ex. 5: Let $P = \{x, y, z\}$

where $X = \{\ldots, -3, 0, 3, 6, \ldots \}$
$Y = \{\ldots, -2, 1, 4, 7, \ldots \}$
$Z = \{\ldots, -1, 2, 5, 8, \ldots \}$

be our partition of $\mathbb{Z}$ from before.

- Let $R_P$ be the associated equiv. relation:
  $(x, y) \in R_P$ iff $\exists \varepsilon \in P$
  s.t. $x = s$ ad $y = s$.

- By our theorem, this defines an equivalence relation.
- Easy to see this is the same equiv. relation \( \equiv_2 \) that we defined previously in a different way:

\[ x \equiv_2 y \text{ if } 3 \mid y - x. \]

- Notice: the equiv. classes of the relation are exactly the pieces of the partition.

2: Let \( P = \{ [1,3], [2,7,17] \} \).

- Then \( P \) is a partition of the set \( A = \{1,2,3,9,17\} \) into 2 pieces.

- Let \( R_P \) be the associated equiv. relation.

\[ R_P = \{(1,3), (2,7), (3,17), (4,4), (2,7), (3,2), (2,4), (4,2), (3,4), (4,3)\} \]

- In this case we can actually write \( R_P \) as a set of ordered pairs, explicitly, in roster notation.
- No real rhyme or reason to this equiv. relation, but still a perfectly good one.

Order Relations

- another common type of binary relation is an order relation
- unlike equiv. relations order relations come in several flavors:
  - nonstrict / strict
  - partial / total.

Def: A relation \( R \) on a set \( A \) is a (nonstrict) partial order iff \( R \) is reflexive, transitive, and antisymmetric.

⇒ If \( R \) is a partial order on \( A \) we say that the pair \( (A, R) \) is a partially ordered set, or poset.

Ex's: \( \leq \) is a partial order on \( \mathbb{R} \). Why? For \( x, y, z \in \mathbb{R} \) we have:

(i) \( x \leq x \) ✓
(ii) if \( x \leq y \) and \( y \leq z \) then \( x \leq z \) ✓
(iii) if \( x \leq y \) and \( y \leq x \) then \( x = y \) ✓
so \((\mathbb{R}, \leq)\) is a poset.

2. Let \(A\) be a fixed set. Then the subset relation \(\subseteq\) on \(\mathcal{P}(A)\) is a partial order.
   
   why: \(\forall x, y, z \in \mathcal{P}(A)\) we have:
   
   (i) \(x \subseteq x\)
   
   (ii) if \(x \subseteq y\) and \(y \subseteq z\) then \(x \subseteq z\)
   
   (iii) if \(x \subseteq y\) and \(y \subseteq x\) then \(x = y\)

\(\therefore\) \((\mathcal{P}(A), \subseteq)\) is a partially ordered set.

3. We showed before that the divisibility relation \(\mid\) on \(\mathbb{N}\) (i.e. \(n \mid m\) iff \(\exists k \in \mathbb{N}\) \(m = nk\)) is reflexive, transitive, and antisymmetric. Hence \((\mathbb{N}, \mid)\) is a poset.

**Question** is \((\mathbb{Z}, \mid)\) a poset?

We still have reflexivity and transitivity. What about antisymmetry?

If \(n \mid m\) and \(m \mid n\) do we have \(n = m\)?

\(\therefore\) Consider 2 and -2

\(2\mid-2\) and \(-2\mid 2\) but \(-2 \neq 2\).
So the divisibility relation 1 on \( \mathbb{Z} \) is reflexive, antisymmetric, hence (II) (i) is not a poset.

These examples of partial orders seem to be of different kinds — and yet, any theorems that can be proved about them using only the properties of reflexivity, transitivity and antisymmetry must be true for all of these! (and any other poset).

**Strict partial orders**

**Defn:** a relation \( R \) on \( A \) is **irreflexive** if \( \forall x \in A \) \((x, x) \notin R\).

E.g. < and \( \neq \) are irreflexive since we never have \( x = x \) or \( x \neq x \).

**Defn** a relation \( R \) on a set \( A \) is called a **strict partial order** if \( R \) is (i) irreflexive (ii) transitive (iii) antisymmetric.

So this is official defn, but by (ii) this is same as saying: \( R \) is antisymmetric and transitive.
Ex. 1. \((\forall x, y \in \mathbb{R}) (x, y) \in R \Rightarrow (y, x) \notin R\).

\[\exists s \quad 1 < s \text{ is a strict partial order on } \mathbb{R}.\]

**Proof:** \(\forall x, y, z \in \mathbb{R}\) we have:

(i) \(x \neq x\)
(ii) \(x < y \text{ and } y < z \Rightarrow x < z\)
(iii) if \(x < y\) and \(y < x\) then \(y = x\)

become always false.

By (iii) instead of checking (i) and (iii) can instead observe:

(iv) \(x < y \Rightarrow y \neq x\).

\(\exists Y \mathbb{A} \text{ be a fixed set. Then } \exists Y \text{ is a strict p.c. on } \mathbb{P}(\mathbb{A}).\)

**Proof:** \(\forall x, y, z \in \mathbb{P}(\mathbb{A})\)

(i) \(x \neq y \text{ and } y \neq z \Rightarrow x \neq z\)
(ii) \(X \neq Y\) then \(Y \cup \{x\} \neq \{x\}\).