Chapter 4: Intro to Mathematical Logic

Goals: learn how to write statements more formally (w/ more symbols, fewer words)
- See how: the form of a statement suggests the form of its proof.

Recall: Def'n (intuitive): A mathematical statement (or prop'n) is a grammatically correct declarative sentence that is true or false.
- May consist of words and/or symbols.
- "Statement" can be rigorously defined, but need more formal logic.
- In that context, "grammatically correct" also has precise meaning.

Ex's
1) Every integer is a real number (T)
2) Every real number is an integer (F)
3. There exists \( x \in \mathbb{R} \) s.t. \( x \notin \mathbb{Z} \) (T).
4. \( 1 + 1 = 2 \) (T)
5. There are infinitely many twin primes (unknown... but either T/F).

Norrex's ① E! IT (grammatically incorrect/meaningless)
② Shakespeare (not declarative/ no truth value)
③ \( x^2 + 1 = 2 \)

→ meaningful sequence of symbols asserting an equality... but no truth value unless \( x \) is specified
- called a variable proposition:
a sentence that becomes a statement once its variables are specified (or quantified over... more on this later)

- we'll use \( P, Q, R, \ldots \) for statements and \( P(x), Q(xy) \)

for var. prop's.

E.g. might say: - let \( P \) denote "\( 5^2 + 1 = 2 \)" (F)
- let \( Q(x) \) denote "\( x^2 + 1 = 2 \)"

Then \( Q(5) \) is the statement \( 5^2 + 1 = 2 \) (F)
and \( Q(1) \) is \( 1^2 + 1 = 2 \) (T)
Merc var. prop's :  
1. $x^2 + 1 \leq 0$
2. $x \in \mathbb{Z}$ and $x^2 < 39$
3. $z = x + y$.

indicate when abbreviating a var. prop'n w/ multiple variables, e.g. could use $Q(x, y, z)$ to denote 3.

Then: $Q(1, 2, 3)$ is F  
but $Q(5, 2, 3)$ is T.

Quantifiers: the other way to turn a var. prop'n into a statement is to quantify over its variables.

E.g. "$x^2 + 1 = 2$" is a var. prop'n  
but "There exists $x \in \mathbb{R}$ s.t. $x^2 + 1 = 2$" is a statement (T)  
as is: "For every $x \in \mathbb{R}$ we have $x^2 + 1 = 2$" (F)

The clauses "There exists $x \in \mathbb{R}$..." and "For every $x \in \mathbb{R}$..." are two types of quantification of the variable $x$. 
- We'll use the symbols:

\[ \forall \quad \exists \]

read: "For all" or "for every"
read: "there exists"

called the "universal quantifier"
called the "existential quantifier"

- Given a prop. \( P(x) \) and a set \( S \), we have that:

"For all \( x \in S \) we have \( P(x) \)"
"there exists \( x \in S \) such that \( P(x) \)"

are statements.

- We denote them by:

\[ (\forall x \in S) \ P(x) \quad (\exists x \in S) \ P(x) \]

(Back way: \( \forall x \in S. \ P(x) \)
\( \exists x \in S. \ P(x) \))

respectively.

Ex's:

1. \( (\exists x \in \mathbb{N}) \ (x<5) \)
   read: "there exists \( x \in \mathbb{N} \) s.t. \( x<5 \)" (T)
2. \( (\forall x \in \mathbb{N}) \ (x<5) \)
   "For every \( x \in \mathbb{N} \), we have \( x<5 \)" (F)
3. \( (\forall x \in \mathbb{N}) \ (x>0) \) (T)
4. \( (\forall x \in \mathbb{R}) \ (x>0) \) (F)
Multiple quantifiers:

3) \((\forall x, y \in \mathbb{N})(x + y \geq 2)\)
   
   read: "For all \(x\) and \(y\) in \(\mathbb{N}\), we have \(x + y \geq 2\)" (T)

   - can also nest \(\forall\)'s and \(\exists\)'s, but beware: order of quantifiers is important!

6) \((\forall x \in \mathbb{N})(\exists y \in \mathbb{R})(y^2 = x)\)
   
   "For every \(x \in \mathbb{N}\) there is a \(y \in \mathbb{R}\) s.t. \(y^2 = x\)"

   i.e. every natural number has a real square root (T)

7) \((\forall x \in \mathbb{N})(\exists y \in \mathbb{N})(y^2 = x)\)
   
   i.e. every natural \# has a square root in \(\mathbb{N}\) (F).

   - what happens if we reverse the order of quantifiers in 6?

   get:
   
   \((\exists y \in \mathbb{R})(\forall x \in \mathbb{N})(y^2 = x)\)

   i.e. "there is a real number \(y\) s.t. every natural number is equal to \(y^2\)"
perfectly well-written statement, but absurd and definitely false

- moral order of quantifiers makes a big deal!

it can also have "inside quantifiers" e.g.

8 \((\forall x \in \mathbb{N})(x > 0 \text{ and } (\exists y \in \mathbb{N})(y > x))\)

9 \((\forall x \in \mathbb{R})(\text{if } x > 0, \text{ then } (\exists y \in \mathbb{R})(y^2 = x))\)

are both statements (both T).

Note on quantifying set variables:

- we've insisted all quantified variables range over a specific set,
  e.g. \((\forall x \in \mathbb{R})(x^2 \geq 0)\) is meaningful
  \((\forall x)(x^2 \geq 0)\) is not

- what if we want to quantify over variables referring to sets?
  e.g. to write,

  "For every set \(S\), we have \(\emptyset \subseteq S\)"

  symbolically, might try:

  \((\forall S \in \mathcal{P} \mathfrak{S})(\emptyset \subseteq S)\)

  set of all sets??
Connectives and Truth Tables

- Connectives are symbols used to combine multiple statements into one.
- All our connectives will be binary.
- Except negation which is unary.
- Truth tables tell us how truth of connected statements depends on truth of the original constituents.

Conjunction ("and")

- Conjunction of statements $P, Q$ is written $P \land Q$ ("P and Q").
- $P \land Q$ is true if both $P, Q$ true.
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<table>
<thead>
<tr>
<th>P</th>
<th>Q</th>
<th>P \land Q</th>
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<tbody>
<tr>
<td>T</td>
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<td>T</td>
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<tr>
<td>T</td>
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<td>F</td>
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Ex5: Let \( P \) denote:
\[(\forall x \in \mathbb{Z})(x+1 > x)\]

Let \( Q \) be:
\[97 \text{ is prime}\]

Let \( R \) be:
\[2^2 = 5\]

Then \( P, Q \) are \((T)\) but \( R \) is \((F)\).

Hence \( P \land Q \) is \((T)\)

but \( P \land R \) and \( Q \land R \) are both \((F)\).

Written out, \( P \land Q \) is:
\[(\forall x \in \mathbb{Z})(x+1 > x) \land (97 \text{ is prime})\]

\((\text{inserting parentheses can clarify expression})\)
**Disjunction ("or")**

- Disjunction of \( P, Q \) written \( P \lor Q \)
  ("P or Q")
- \( P \lor Q \) is true if at least one of \( P, Q \)
  is true:

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<thead>
<tr>
<th>( P )</th>
<th>( Q )</th>
<th>( P \lor Q )</th>
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<tbody>
<tr>
<td>( T )</td>
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<td>( T )</td>
<td>( F )</td>
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<td>( F )</td>
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</table>

E.g.: \((\forall x \in \mathbb{R})(x^2 \geq 0) \lor (96 \text{ is prime})\)
  is \( T \), but \((\forall x \in \mathbb{R})(x^2 \geq 0) \lor (96 \text{ is prime})\)
  is \( F \).

**Negation ("not")**

- Only unary connective we’ll use
- Negation of \( P \) written \( \neg P \)
- \( \neg P \) true iff \( P \) is false:

<table>
<thead>
<tr>
<th>( P )</th>
<th>( \neg P )</th>
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<tbody>
<tr>
<td>( T )</td>
<td>( F )</td>
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<td>( F )</td>
<td>( T )</td>
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</table>
Ex 5:  

1. \((\forall x \in \mathbb{N}) (\exists y \in \mathbb{N})(y^2 = x)\)

is \((F)\), hence:

2. \((\forall x \in \mathbb{N}) (\exists y \in \mathbb{N})(y^2 = x)\)

is \((T)\), hence:

3. \(\exists x (\forall y \in \mathbb{N})(y^2 = x)\)

is \((F)\) again.

4. For any statement \(P\), the statement \(P \lor \neg P\) is \((T)\), whereas \(P \land \neg P\) is \((F)\).

E.g. \((96 \text{ u prime}) \lor (96 \text{ u prime})\) is \((T)\), but \((96 \text{ u prime}) \land (96 \text{ u prime})\) is \((F)\).

We can use connectives in var. prop/Prop.

E.g. \(\exists x \forall y \exists z (x^2 + y^2 = z^2)\)

then \(P(3,5)\) is true
while \(P(3,6)\) is false.

and \((\exists x, y \in \mathbb{N}) P(x, y)\) is true
while \((\forall x, y \in \mathbb{N}) P(x, y)\) is false.

We can also use in def/defns, set-builder notation etc.
- e.g. If $A, B$ are subsets of a universal set $U$, then:

$$A \cap B = \{x \in U \mid x \in A \land x \in B\}$$

$$A \cup B = \{x \in U \mid x \in A \lor x \in B\}$$

$$\overline{A} = \{x \in U \mid \neg (x \in A)\}$$

- We'll explore connections between set operations and connectivity more later.

**Implication:**

- Given statements $P, Q$, the statement $P \implies Q$ is read "$P$ implies $Q$" or "If $P$, then $Q$."  
- $P \implies Q$ is true if whenever $P$ is true, $Q$ is also true.

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<thead>
<tr>
<th>$P$</th>
<th>$Q$</th>
<th>$P \implies Q$</th>
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<tbody>
<tr>
<td>$T$</td>
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- Notice: $P \implies Q$ is always true when $P$ is false, which is often confusing when first learning the connective.
- $P \implies Q$ is only $(F)$ when $P$ is $(T)$ but $Q$ is $(F)$.  

\[\]
Statements of form \( P \Rightarrow Q \) are called conditional statements.

Ex. 5:

1. "\( 1 + 1 = 2 \Rightarrow (1 + 1) + 1 = 3 \)" is true.
2. "\( 1 + 1 = 2 \Rightarrow (1 + 1) + 1 = 4 \)" is false.
3. "\( 1 + 1 = 2 \Rightarrow \sqrt{2} \notin \mathbb{N} \)" is true even though the \( P \) and \( Q \) in this example are not apparently related statements.
4. "My name is Sally \( \Rightarrow \) my name begins with S." is true.

- Both the premise \( P \) and conclusion \( Q \) in this case are (F), but the conditional \( P \Rightarrow Q \) is true.

Illustration: "Any false implies false" is true.

5. "Tomorrow is Sunday \( \Rightarrow \) my name is Garrett" is also true.
   ("false \( \Rightarrow \) true" is true)

6. \( (\exists x \in \mathbb{R})(x^2 = -1) \Rightarrow 1 + 1 = 3 \) is true! Automatically since premise \( (\exists x \in \mathbb{R})(x^2 = -1) \) is false, despite fact it has no apparent relation to conclusion.
7. can also use \( \Rightarrow \) in var prop's

E.g.,

\[ x \geq 2 \Rightarrow x^2 \geq 4 \]

is a well-formed var. prop'n and

\[ (\forall x \in \mathbb{R}) (x \geq 2 \Rightarrow x^2 \geq 4) \]

is \underline{true}, because:

For every \( x \in \mathbb{R} \), either \( x \geq 2 \), in which case \( x^2 \geq 4 \) (i.e., \( x \geq 2 \Rightarrow x^2 \geq 4 \)) holds because "true \Rightarrow true" is \underline{true}; or \( x < 2 \) in which case \( x \geq 2 \Rightarrow x^2 \geq 4 \) holds automatically (because "false \Rightarrow ..." is \underline{true}).

8. \( \mathbf{OTOL}: (\forall x \in \mathbb{R}) (x^2 \geq 4 \Rightarrow x \geq 2) \)

is false because there is a real number \( x \) (e.g., \( x = -3 \)) such that

\[ x^2 \geq 4 \ \underline{\text{true, but}}\]
\[ x \geq 2 \ \underline{\text{false}}. \]
Equivalence

Given statements \( P, Q \), the statement \( P \iff Q \) (read: "\( P \) if and only if \( Q \)"

or: "\( P \iff Q \)"

is true if and only if \( P, Q \) have the same truth value.

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<thead>
<tr>
<th>( P )</th>
<th>( Q )</th>
<th>( P \iff Q )</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
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</tbody>
</table>

\[ \text{ex'}: \ G \quad (1+1 = 2) \iff (1+1+1 = 3) \quad \text{u} (T) \]

\[ \text{ex'}: \quad (1+1 = 3) \iff (1+1+1 = 4) \quad \text{u} (T) \]

\[ \begin{align*}
\text{ex'}: \quad (\forall x \in \mathbb{N})(n>0) \iff (1+1 = 2) \quad \text{u} (T) \\
\text{ex'}: \quad (1+1 = 2) \iff (2+2 = 5) \quad \text{u} (F) \\
\end{align*} \]

\[ \text{ex'}: \quad (\forall x \in \mathbb{R})(x>0) \iff (\exists y \in \mathbb{R})(y^2=x^2) \quad \text{u} (T) \]

\[ \text{why}: \quad \text{For any fixed} \ x_0 \ \text{in} \ \mathbb{R}, \ 
\text{the statements} \ "x_0>0" \ \text{and} \ "(\exists y \in \mathbb{R})(y^2=x_0)" \ 
\text{are either both true, or both false.} \]
**Definition**: Two statements $P, Q$ are said to be **logically equivalent** if $P \iff Q$ is true.

- E.g., $1+1=2$ and $1+1+1=3$ are logically equiv.
- We're more interested in **logically equiv. forms** for connected (e.g., negated) and **quantified statements**.

**Negating Quantified Statements**:

- $P(x)$ is a var. prop., and $S$ is a set.
- Consider the negated statements:

\[
\begin{align*}
\neg 1 & \,(\forall x \in S) P(x) \\
\neg 2 & \,(\exists x \in S) P(x)
\end{align*}
\]

**Observe**:

\[
\begin{align*}
& \neg 1 \text{ is true iff there is an } x \in S \\
& \text{ s.t. } P(x) \text{ is false, i.e. iff } \\
& (\exists x \in S) \neg P(x) \text{ is true}.
\end{align*}
\]

\[
\begin{align*}
& \neg 2 \text{ is true iff for every } x \in S \text{ we have } P(x) \text{ is false, i.e. iff } \\
& (\forall x \in S) \neg P(x) \text{ is true}.
\end{align*}
\]
Thus shows:

\[ \neg (\forall x \in S) P(x) \iff (\exists x \in S) \neg P(x) \]

is always true (regardless of the prop'ns \( P(x) \))
i.e. that \( \neg (\forall x \in S) P(x) \) and \( (\exists x \in S) \neg P(x) \) are logically equiv.

Likewise:

\[ \neg (\exists x \in S) P(x) \iff (\forall x \in S) \neg P(x) \]

is always true.

\[ \rightarrow \] these equivalences are often useful when trying to prove quantified statements by contradiction.

Ex's:

\[ \neg (\forall x \in \mathbb{R}) (x \in \mathbb{N}) \]

is equiv. to:

\[ (\exists x \in \mathbb{R}) \neg (x \in \mathbb{N}) \]

"not all reals are naturals"

"there is a real which is not a natural"

(Note: we'll often write \( \neg (x \in \mathbb{N}) \) as \( x \notin \mathbb{N} \), \( \neg (x=y) \) as \( x \neq y \), etc...)

\( \neg (\exists x \in \mathbb{R}) (x+1 = 0) \)

is equiv. to:

\( (\forall x \in \mathbb{R}) (x+1 \neq 0) \)

"there is no additive inverse to 1 in \( \mathbb{R} \)"

"every real is not an additive inverse for 1"

(In this case, both statements are false)
For multiple quantifiers, just iterate the process...

\[ \forall x \in \mathbb{R} \quad \exists y \in \mathbb{R} \quad (xy = 1) \rightarrow \text{"not every real has a multiplicative inverse"} \]
\[ \forall x \in \mathbb{R} \quad \exists y \in \mathbb{R} \quad (xy = 1) \]
\[ \forall x \in \mathbb{R} \quad \forall y \in \mathbb{R} \quad (xy \neq 1) \rightarrow \text{"there is a red w/o a multiplicative inverse"} \]

In this case: all are true since \( x = 0 \) has no inverse.

Negating connected statements

Prop'n: For any statements \( P, Q \) the following logical equivalences hold (i.e., the following statements are always true):

1. \( \neg \neg P \equiv P \)
2. \( \neg (P \land Q) \equiv \neg P \lor \neg Q \)
3. \( \neg (P \lor Q) \equiv \neg P \land \neg Q \)

PF: To prove, we'll use truth tables.

<table>
<thead>
<tr>
<th>( P )</th>
<th>( \neg P \lor \neg P )</th>
<th>( \neg \neg P \lor \neg P )</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
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\( \neg \neg P \lor \neg P \) is always true.