Hence \( |S| = (\binom{n-1}{k-1}) + (\binom{n-1}{k}) \)
\[ \text{i.e. } (\binom{n}{k}) = (\binom{n-1}{k-1}) + (\binom{n-1}{k}) \checkmark \]

Claim. Fix \( n, k \in \mathbb{N}, \ k \leq n. \) Then
\[ (\binom{n}{k}) = n(\binom{n-1}{k-1}) \]

Proof: Let \( S \) denote the set of committees of \( k \) people from a group of \( n \) people that include a specified chairperson. Such a committee can be formed by:
- picking the committee members \( (\binom{n}{k}) \)
- from there, picking the chairperson \( (\binom{k}{1}) = k \)

\[ \text{hence } |S| = (\binom{n}{k})k \]
or could form a committee by:

- picking a chairperson first \( (\binom{n}{1}) \)
- from remaining \( n-1 \) people pick remaining \( k-1 \) committee members \( \binom{n-1}{k-1} \)

\[
\text{hence } \binom{n}{1} = \binom{n}{k-1} \text{ too}
\]

\[
\text{hence } \binom{n}{k-1} = \binom{n}{k} \text{ true.}
\]

In this case, can verify the identity algebraically:

\[
\binom{n}{k} \cdot k = \frac{n!}{k!(n-k)!} \cdot k = \frac{n!}{(k-1)!(n-k)!}
\]

\[
\binom{n-1}{k-1} = \binom{n}{k-1} \frac{(k-1)!}{(k-1)!(n-k-1)!}
\]

\[
= \frac{n!}{(k-1)!(n-k)!} \text{ same}
\]
Propn: Fix \( n \in \mathbb{N} \). Then:
\[
n \cdot 2^{n-1} = \sum_{k=1}^{n} \binom{n}{k}
\]

PF: Let \( S \) be the set of nonempty committees w/ a chairperson chosen from a group of \( n \) people.

\( \text{Done} \sqrt{\text{v}} \)

But CAT's check:
- can form committee by:
  - choosing the chair \( \binom{n}{1} = n \)
  - from remaining \( n-1 \), choosing other committee members
    (i.e. just choosing a subset from a set of size \( n-1 \))

\[
\Rightarrow |S| = n \cdot 2^{n-1}
\]

\( \text{\textendash\textendash\textendash\textendash\textendash\textendash\textendash\textendash} \) or can partition \( S \):
\[
S = A_1 \cup A_2 \cup \ldots \cup A_n
\]
where \( A_k \) is the set of committees of exactly \( k \) people,
above we computed
\[
|A_k| = \binom{n}{k} k
\]
Hence
\[
|S| = |A_1| + |A_2| + \ldots + |A_n|
= \binom{n}{1} 1 + \binom{n}{2} 2 + \ldots + \binom{n}{n} n
= \sum_{k=1}^{n} \binom{n}{k} k
\]

As a bonus, using previous example, could re-write this identity as:
\[
2^n - 1 = \sum_{k=1}^{n} \binom{n}{k} k = \sum_{k=1}^{n} n \binom{n-1}{k-1}
\]

**Inclusion / Exclusion:**

Ros says: if we write a set, \( A \) as:
\[
A = A_1 \cup A_2 \cup \ldots \cup A_k
\]
and then \( A_i \)'s are disjoint, then
\[|A_1| = |A_1| + |A_2| + \ldots + |A_k| .\]

... but what if the \(A_i\)'s are not disjoint? Can we still count \(|A_1|\) in terms of the \(|A_i|\)'s?

Yes, but we need the principle of inclusion/exclusion.

**Ex:** Let \(A_1 = \{\ast, 0, 1, 3\}\)
\[A_2 = \{\Delta, 0, \sim\}\]

\(A = A_1 \cup A_2\)

What is \(|A_1|\)?

\[
\begin{array}{c}
\ast \ast \\
\heartsuit D \hfill O \hfill \sim \\
\end{array}
\]

\(A_1, A_2\)

Is \(|A| = |A_1| + |A_2|\)? Not quite — since \(\Delta \in A_1, \Delta \in A_2\).
but we can think of counting $|A_1|$ as $|A_1| + |A_2| \ldots$ then correcting over-counting.

In this case we count the elements in $A_1 \cap A_2$ (i.e. $0$) twice, hence

$$|A_1| = (|A_1| + |A_2| - |A_1 \cap A_2|)$$

$$= 3 + 3 - 1$$

$$= 5$$

and indeed $A = A_1 \cup A_2 = \{0, 1, 2, 3, 5\}$

This is general!

For any finite set $A$, if $A = A_1 \cup A_2$
then $|A| = |A_1| + |A_2| - (A_1 \cap A_2)$

What about 3 sets?

Ex: sets $A_1 = \{0, 1, 2, 3, 5\}$
$A_2 = \{1, 2, 3, 5\}$
$A_3 = \{0, 1, 2, 3, 5\}$
\[ |A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_3| + |A_1 \cap A_2 \cap A_3| \]

el's here counted twice

el's here counted twice, then subtracted three times! So need to add back!
Reasoning more generally like this yields:

**Theorem (Principle of Inclusion/Exclusion)** (PIE)

If $A \cup A_1 \cup \cdots \cup A_k$ is finite and

$$A = A_1 \cup \cdots \cup A_k$$

then: $|A| = |A_1 \cup \cdots \cup A_k|$

$$= \sum_{i=1}^{k} |A_i| - \sum_{1 \leq i < j \leq k} |A_i \cap A_j| + \sum_{1 \leq i < j < k \leq k} |A_i \cap A_j \cap A_k| - \cdots + (-1)^{k-1} |A_1 \cap \cdots \cap A_k|$$

$\left(\text{each term}\right)$
Ex C 65s in a group of 50 ppl

- 30 have brown hair
- 25 have blue eyes
- 10 have both

(i) How many have either brown hair or blue eyes (or both)?
(ii) How many have neither?

Sory: Let $A_{\text{brown}} = \text{set of brown hair}$

$A_{\text{blue}} = \text{set of blue eyes}$

Then, by PIE:

(i) $|A_{\text{brown}} \cup A_{\text{blue}}| = |A_{\text{brown}}| + |A_{\text{blue}}| - (A_{\text{brown}} \cap A_{\text{blue}})$

$= 30 + 25 - 10$

$= 45$

(ii) So # w/ neither $= 50 - 45 = 5$
Ex 2: How many integers between 0 and 1000 are not divisible by any of the numbers 5, 7, 11?

Set \( A_5 \) = set of integers between 0 and 1000 divisible by 5

So \( A_5 = \{5, 10, 15, \ldots, 1000\} \)

\[ |A_5| = \left\lfloor \frac{1000}{5} \right\rfloor = 200 \]

Set \( A_7 \) = set of integers between 0 and 1000 divisible by 7

\[ A_7 = \{7, 14, \ldots, 999\} \]

\[ |A_7| = \left\lfloor \frac{1000}{7} \right\rfloor = 142 \]

Set \( A_{11} \) = set of integers between 0 and 1000 divisible by 11

\[ |A_{11}| = \left\lfloor \frac{1000}{11} \right\rfloor = 90 \]

Now: \( A_5 \cap A_7 = \) set of \( n \leq 1000 \) divisible by both 5, 7

\[ = \emptyset \]

\( = A_{35} \)
hence \(|A_5 \cap A_7| = |A_{35}| = \left\lfloor \frac{1000}{35} \right\rfloor = 28\)

Similarly: \(|A_5 \cap A_{11}| = |A_{55}| = \left\lfloor \frac{1000}{55} \right\rfloor = 18\)

\(|A_7 \cap A_{11}| = |A_{77}| = \left\lfloor \frac{1000}{77} \right\rfloor = 12\)

Finally, \(|A_5 \cap A_7 \cap A_{11}| = |A_{355}| = \left\lfloor \frac{1000}{555} \right\rfloor = 2\)

So, by PIE:
\(|A_5 \cup A_7 \cup A_{11}| = |A_5| + |A_7| + |A_{11}| - (|A_5 \cap A_7| + |A_5 \cap A_{11}| + |A_7 \cap A_{11}| - |A_5 \cap A_7 \cap A_{11}|)\)
\[ = 200 + 142 + 90 - 28 - 0 - 12 + 2 \]
\[ = 376 \]

We're interested in the complement of this set.

\[ \Rightarrow \text{\# of } n \leq 1000 \text{ divisible by } \text{none of } 5, 7, 11 = 1000 - 376 \]
\[ = 624 \]