Countin' Poker hands.

- A standard deck consists of 52 cards.
- Each card has 1 of 4 suits (♦, ♣, ♠, ♦) and 1 of 13 ranks: (A, 2, 3, 4, ..., 9, 10, J, Q, K).
- E.g., A♦ and 2♠ are cards.
- A poker hand is a 5-selection from a standard deck.

Ex 6 How many distinct hands are possible?

\[ \binom{52}{5} = 2,598,960 \]

- A full house is a hand consisting of 3 cards of one rank, and 2 cards of another, e.g., A♦, A♣, 3♦, 3♣, 3♦.

Q: How many distinct full house hands are possible?
Solution:
- Pick two ranks \((13)\)

- From these, pick the 3-card rank \((\frac{4}{3})\)

- Pick three cards from this rank \((\frac{4}{3})\)

- Pick two cards from the 2-card rank \((\frac{4}{3})\)

\[= \binom{13}{1} \binom{12}{3} \binom{4}{1} \binom{3}{2} = 3,749.\]

\(3\) A 3-of-a-kind consists of 3 cards from a single rank and 2 other cards from two other distinct ranks, e.g., 3 aces, 2 10s, and a king.

Q: How many 3-of-a-kind hands are possible?

Solution:
- Pick 3-card rank \((\frac{13}{1})\)

- From this rank, pick 3 cards \((\frac{4}{3})\)
- Pick remaining two ranks \( \binom{12}{2} \)
- From the first of these, pick a card \( \binom{4}{1} \)
- And from the second \( \binom{4}{1} \)

\[ \Rightarrow \text{# of 3-of-a-kinds is} \]

\[ \binom{13}{1} \binom{4}{1} \binom{12}{2} \binom{4}{1} \binom{4}{1} = 599912 \]

Alt Sol'n:
- Pick three ranks \( \binom{3}{3} \)
- From these, pick the 3-card rank \( \binom{3}{3} \)
- From this rank, pick 3 cards \( \binom{4}{3} \)
- From other two ranks, pick cards \( \binom{4}{1} \binom{4}{1} \)

But: \( \binom{13}{3} \binom{4}{1} \binom{12}{1} \binom{4}{1} \binom{4}{1} = 599912 \) too \( \checkmark \)
Binary sequences

- A binary sequence (of length \( n \)) is an ordered sequence of 0's and 1's (of length \( n \)).

  - E.g., \( s = 011 \) is a binary sequence of length 3.

- Let \( P_n \) denote the set of all binary sequences of length \( n \).

**Ex.**

1. What is \( |P_n| \)?
2. How many sequences \( s \in P_n \) have at least two 1's?

**Sol'n.**

1. Each \( s \in P_n \) formed by making \( n \) choices.

\[
\begin{array}{c|c|c}
 0/ & 1/ & \cdots \\hline
 0/ & 1/ & \cdots \\
\end{array}
\]

- 2 choices each

\[ |P_n| = 2^n \]

2. Easier to count # of seq's w/ zero 1's or one 1 and subtract from total.
\* w/ zero 1's = 1  (just 11-1)
\* w/ one 1 = n  (one for each place to put the 1)
so \* w/ at least two 1's

= \(2^n - n - 1\) ✔

**Theorem** Fix \(n \geq 0\)

then: \(2^n = \binom{n}{0} + \binom{n}{1} + \ldots + \binom{n}{n}\)

= \(\sum_{k=0}^{n} \binom{n}{k}\)

**Proof:** if \(n = 0\), then \(2^0 = 1 = \binom{n}{0} = \sum_{k=0}^{0} \binom{n}{k}\)

so sps \(n \geq 1\).

we proved \(|P_n| = 2^n\)

we can partition \(P_n = S_0 \cup S_1 \cup \ldots \cup S_n\)

where \(S_k\) = set of sequences w/ exactly \(k\)-many 1's.

observe: \(|S_k| = \binom{n}{k}\) \(\rightarrow\) # of ways to pick \(k\) positions when 1's appear

Hence: \(2^n = |P_n| = |S_0| + |S_1| + \ldots + |S_n|\)

= \(\binom{n}{0} + \binom{n}{1} + \ldots + \binom{n}{n}\)

= \(\sum_{k=0}^{n} \binom{n}{k}\) ✔
Selections w/ repetition:

Q: How many ways to select a object from $k$ types of objects, if repetition is allowed?

Ex: Dee's donuts sells 4 types of donuts, and you want to buy a dozen donuts, how many distinct ways to do this?

Sol'n: Imagine putting down 3 "spacers" to separate donut types:

```
  ooo | ooo | ooo | oo
```

<table>
<thead>
<tr>
<th>4</th>
<th>3</th>
<th>2</th>
</tr>
</thead>
</table>

Type I | II | III |

So would the above "donut + spacer" diagram correspond to a purchase of:

- 3 donuts of type I
- 2 of type II
- 5 of type III
- 2 of type IV

We can view such a diagram as a sequence: w/ 12 o's (for a dozen donuts) 3 i's (to separate 4 types)
Conversely any such sequence \((120's 31's)\) corresponds to a selection of dominoes.

- e.g. \(10100000000010000\)
corresponds to a selection of
  
  \[
  \begin{align*}
  &0 \quad \text{tp I domino} \\
  &1 \quad \text{tp II domino} \\
  &7 \quad \text{tp III domino} \\
  &4 \quad \text{tp IV domino}
  \end{align*}
  \]

- Hence: \# of ways to make a selection
  
  \[
  = \# \text{ of binary seqs of length 15 w/}
  \]
  
  three 1's

  \[
  = \binom{15}{3} = 455. \checkmark
  \]

Same reasoning in general proves:

**Theorem** The \# of ways to select \(n\) objects from \(k\) types w/ repetition allowed is

\[
\binom{n+(k-1)}{k-1}
\]

\((k-1)\) because we only need \(k-1\) "spacers" for separate \(k\) types of objects.
Ex. 5ps we roll a (indistinguishable) 6-sided dice.

How many distinct outcomes are possible?

Solv'n: each of the n dice can roll into 6 possible "types"

\[
\begin{array}{cccccc}
1 & 2 & 3 & 4 & 5 & 6 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\end{array}
\]

hence \( \# \) of possible outcomes is:

\[
\binom{n+(6-1)}{6-1} = \binom{n+5}{5}
\]

So if we roll 10 dice \( \# \) is:

\[
\binom{15}{5} = 3003 \checkmark
\]

Ex: How many anagrams of the word "LIMITING" are there?

Solv'n: Two approaches: 1) First distinguish the E's w/ subscripts \( \tilde{1}, \tilde{2}, \tilde{3} \).
- # of anagrams w/ distinguished
  I's is just 8!
  For each anagram w/ distinguished
  I's there are 3! equivalent anagrams
  w/ I's not distinguished
  (one for each permutation of I_1I_2I_3)
  \[ \implies \text{# of anagrams} = \frac{8!}{3!} \]

2) Can think of anagram being formed in two stages:
   (i) pick 3 positions for the I's \( \binom{8}{3} \)
   (ii) for remaining 5 positions pick an ordering of LMTNG: 5!
   \[ \implies \text{# of anagrams} = \binom{8}{3} \cdot 5! \]
   \[ = \frac{8!}{3!5!} = \frac{8!}{3!} \]
   \[ = 6720, \text{ as before.} \]

**Countin' in Two Ways:**

\( \text{Thus (Pascal's Identity)} \)

\[ \begin{align*}
\text{w/ } & \text{ w/ KSH}.
\end{align*} \]
Then:
\[
\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}
\]

Proof: Let \( S \) be the set of \( k \) element subsets of \( \{1,2,\ldots,n\} \). Then \( |S| = \binom{n}{k} \).

Observe that we can partition \( S \) into \( S_1 \) and \( T \) where:

- \( S_1 \) is the set of \( k \) element subsets of \( \{1,2,3,\ldots,n\} \) that contain 1.
- \( T \) is the set of \( k \) element subsets of \( \{1,2,3,\ldots,n\} \) that do not contain 1.

Then \( |S| = |S_1| + |T| \).

- Subsets in \( S_1 \) are formed by selecting \( k-1 \) elements from \( \{2,3,\ldots,n\} \) (\( 1 \) automatically included).

\[ |S_1| = \binom{n-1}{k-1} \]

- Subsets in \( T \) are formed by selecting \( k \) elements from \( \{2,3,\ldots,n\} \).

\[ |T| = \binom{n-1}{k} \]
Hence \( ISI = (\binom{n-1}{k-2}) + (\binom{n-1}{k}) \)

i.e. \( \binom{n}{k} = (\binom{n-1}{k-1}) + (\binom{n-1}{k}) \)