Infinity

The concept of cardinality:

- we would say the set \( \{ *, 0, \Delta 3 \} \) has 3 elements, or is of size 3.

- why? By counting it!

\[
\begin{array}{ccc}
* & 0 & \Delta 3 \\
1 & 2 & 3 \\
\end{array}
\]

- in doing so, we are implicitly defining a bijection between the sets \( \{1, 2, 3\} \) and \( \{ *, 0, \Delta 3 \} \)

- we could've counted differently:

\[
\begin{array}{ccc}
* & 0 & \Delta 3 \\
2 & 3 & 1 \\
\end{array}
\]

- generalizing this idea: we'll say two sets have the same size if there is a bijection between them
Def'n: We say that two sets $A, B$ have the same cardinality, and write $A \sim B$, iff there is a bijection $f: A \rightarrow B$.

Note: In set theory courses, one defines, for every set $A$, the cardinal number $|A|$. One can then prove: $A \sim B$ iff $|A| = |B|$. (e.g., $|\mathbb{Z}, *, \cdot, 1| = |\mathbb{Q}, 0, 1| = 3$)
- Defining cardinal #s beyond our scope: for us, $|A| = |B|$ just means $A \sim B$, i.e.,
  $\exists f: A \rightarrow B$ a bijection.

Properties of $\sim$:
1. $\sim$ For any set $A$,
   $\text{id}_A: A \rightarrow A$ is a bijection (why?)
   - Hence $A \sim A$, i.e., $\sim$ is reflexive

2. If $f: A \rightarrow B$ is a bijection, then $f$ is invertible and $f^{-1}: B \rightarrow A$ is a bijection.
   - Hence $\sim$ if $A \sim B$, then $B \sim A$, i.e., $\sim$ is symmetric
On how you showed: if $f: A \to B$ and $g: B \to C$ are bijections, then so $g \circ f: A \to C$ is a bijection. Hence, if $A \sim B$ and $B \sim C$, then $A \sim C$, i.e. $\sim$ is transitive.

So $\sim$ is an equivalence relation on sets!

It is most interesting when the sets being compared are infinite.

Exs: 1. $\{1, 2, 3\} \sim \{2, 0, 3\}$ and $f: \{1, 2\} \to \{2, 0, 3\}$ is a bijection.

2. Let $-N$ denote the set $\{-1, -2, -3, \ldots\}$ and define $f: N \to -N$ where $f(n) = -n$.

\[ f(1) = -1, f(2) = -2, \ldots \]

to check: $f$ is a bijection. Hence $N \sim -N$.

3. Last time we proved: $f: \mathbb{Z} \to N$

defined by

\[ f(n) = \begin{cases} 2n & n > 0 \\ 2(n+1) + 1 & n \leq 0 \end{cases} \]

is a bijection. Hence $\mathbb{Z} \sim N$. 

\[ \begin{array}{c} \text{1} \\ \text{2} \\ \text{3} \end{array} \]
Combining (2) and (3) gives: \( \mathbb{Z} \cong \mathbb{N} \). by transitivity of \( \cong \).

**Defn:** Let \( A, B \) be sets. (or \( |A| \leq |B| \))

1. We write \( A \preceq B \) to mean: there is an injection \( f: A \to B \). (or \( |A| \geq |B| \))
2. We write \( A \preceq B \) to mean: there is a surjection \( g: A \to B \).

\( \Rightarrow \) we'll write \( A \prec B \) to mean: there is an injection \( f: A \to B \) but no bijection \( g: A \to B \) (i.e. \( A \preceq B \) but \( A \not\cong B \)).

\( \Rightarrow \) similarly for \( A \succ B \).

**NOTE!** \( A \preceq B \) is not the "reverse of" \( A \preceq B \), i.e. it is not asserting there is an injection from \( B \) to \( A \).

**But this follows:**

**Theorem:** For all sets \( A, B \) we have:

\[ A \preceq B \text{ iff } B \succeq A \]

\( \text{i.e. there is an injection } f: A \to B \text{ iff there is a surjection } g: B \to A. \)
Before proving theorem, let's do some examples to illustrate ideas:

**Example:** Consider $f: \{1,2,3\} \rightarrow \{1,2,3,4,5\}$ defined by: $f(1) = 1$, $f(2) = 3$, $f(3) = 4$

Observe: $f$ is an injection. Hence $|\{3\}| \leq |\{5\}|$.

Idea: to get a surjection $g: \{5\} \rightarrow \{3\}$ we "reverse" $f$ then map anything left over to something arbitrary: e.g.

$$
\begin{align*}
  g(1) &= 2 \\
  g(3) &= 1 \\
  g(3) &= 3
\end{align*}
$$

Then: $g$ is a function since $f$ was injective and

we clearly surjective.

Can use this idea to prove more generally

if $A \subseteq B$ then $B \supseteq A$. 
Consider: \( g: [5] \rightarrow [3] \) defined by:

\[
\begin{align*}
g(1) &= g(2) = 1 \\
g(2) &= g(5) = 2 \\
g(5) &= 3.
\end{align*}
\]

Observe: \( g \) is a surjection hence \([5] \simeq [3]\).


we take some "reverse" of \( g \) w/o repeats.

e.g. define: \( f: [3] \rightarrow [5] \) by

\[
\begin{align*}
f(1) &= 1, \\
f(2) &= 3, \\
f(3) &= 5.
\end{align*}
\]

"reverses" \( g \) w/o repeats

Then: \( f \) is a function because \( g \) was surjective and w/o repeats because we deleted repeats. It's injective because \( f(1) = f(2) \).

Some idea can be used to prove:

\( B \subseteq A \) then \( A \approx B. \)
Let's now give a formal proof of the theorem:

**Pf** (⇒) Supp A ≤ B, i.e. $\exists f: A \rightarrow B$ an injection. We wish to show $g: B \rightarrow A$ a surjection i.e. $B \supseteq A$.

Define $g$ as follows:
- First, fix some $a \in A$.
- Consider a fixed $b \in B$.

If $b \in \text{Im} f$, then $\exists a \in A$ s.t. $f(a) = b$.
Moreover, this $a$ is unique, since $f$ is injective.

In this case, define $g(b) = a$.

If $b \notin \text{Im} f$, define $g(b) = a_0$.

Then:
1. $g$ is a function from $B$ to $A$.
2. If $b \in \text{Im} f$, then $g(b)$ is unique $a \in A$ s.t. $f(a) = b$.
3. If $b \notin \text{Im} f$, then $g(b) = a_0$.

$g$ is surjective.

[Fix $a \in A$. Let $b = f(a)$, then by def. $g(b) = a$.]

Picture:

![Diagram](image.png)
(⇐) Suppose \( B \supseteq A \), i.e., there is a surjection \( g : B \to A \). We wish there is an injection \( f : A \to B \).

Define \( f \) as follows:

- For a given \( a \in A \), since \( g \) is surjective there is at least one \( b \in B \) s.t. \( g(b) = a \), i.e., \( \text{PreIm}(g(a)) \neq \emptyset \).

- So pick one distinguished element \( b_a \in \text{PreIm}(g(a)) \) and define \( f(a) = ba \). (Observe \( g(b_a) = a \))

- Do this for every \( a \in A \).

Then:
1. \( f \) is a function from \( A \) to \( B \):
   - Every \( a \in A \) has a unique output \( b_a \in B \)
   - From \( \text{PreIm}(g(a)) \)

2. \( f \) is injective: if \( f(a) = f(a') = b \)
   - Then \( g(b) = a \) hence \( c = a' \) since \( g \) is a function.

Picture

\[ \begin{array}{c}
\text{B} \uparrow \downarrow \text{A} \\
\text{g} & & \text{f} \\
\text{A} \uparrow \downarrow \text{B} \\
\text{a} & & \text{b} \\
\text{c} & & \text{a'}
\end{array} \]
Properties of $\leq$ and $\geq$

Suppose $A, B, C$ are sets.

1. $A \leq A$ and $A \geq A$ since $\text{id}_A: A \rightarrow A$ is both an injection and surjection. Hence $\leq$ and $\geq$ are reflexive.

2. If $A \leq B$ and $B \leq C$ then $\exists f: A \rightarrow B$ and $\exists g: B \rightarrow C$ injection. By how, $g \circ f: A \rightarrow C$ is an injection. Hence $A \leq C$.

Similarly if $A \geq B \geq C$ then $A \leq C$.

Hence $\leq$ and $\geq$ are transitive.


If $A \leq B$ and $B \leq A$ not necessarily that $A = B$.

e.g. $A = \{1,2,3\}$ $B = \{*, \emptyset, 0\}$

But!! we'll show in this case that $A \cap B$ always.

4. Are they total? That is, for any two sets $A, B$ do we always have $A \leq B$ or $B \leq A$?

Yes! If you assume axiom of choice (but this beyond scope).
Notation: \(|A1 = 1B|\) means \(A \cap B\)

may also write:

\(|A1 \leq 1B|\) for \(A \subseteq B\)

\(|A1 \geq 1B|\) for \(A \supseteq B\).

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Some paradoxes of infinity

Theme: \(A \cap B\) when \(A, B\) are infinite can be counterintuitive.

\(\square\) Let \(E = \{2, 4, 6, \ldots, 3\}\)

Then \(N \supseteq E\) ("there are as many even numbers as whole numbers")

why: \(F: N \rightarrow E\) defined by \(f(n) = 2n\) is a bijection:

<table>
<thead>
<tr>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
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</thead>
<tbody>
<tr>
<td></td>
<td>2</td>
<td>4</td>
<td>6</td>
<td>8</td>
<td>10</td>
</tr>
</tbody>
</table>

\(\square\) Let \(S = \{1, 9, 9, \ldots, 3\}\) = set of squares

then \(N \cap S\) ("there are as many squares as whole numbers")
Why: \( f: \mathbb{N} \to \mathbb{S} \) defined by \( f(n) = n^2 \) is a bijection.

(3) We have \([0, 1] \sim [0, 2]\) ("there are as many numbers between 0 and 1 as between 0 and 2")

Why: \( f: [0, 1] \to [0, 2] \) defined by \( f(x) = 2x \) is a bijection.

\[\begin{array}{c}
0 & \quad \mathcal{S} \quad \mathcal{S}
\end{array}\]

(4) In fact: "The side is as large as the square" i.e. \([0, 1] \sim [0, 1] \times [0, 1] \)
(i.e. there is a bijection \( f: [0, 1] \to [0, 1] \times [0, 1] \))

Why: beyond our scope.

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Moral: infinite sets are wild!