Schematically:

\[ \text{PreImages:} \]

**Defn:** Sps \( f: A \to B \) is a function and \( Y \subseteq B \). The **preimage** of \( Y \), denoted \( \text{PreIm}_f(Y) \), is defined as:

\[
\text{PreIm}_f(Y) = \{ x \in A \mid f(x) \in Y \}
\]

= the inputs in \( A \) whose outputs are in \( Y \).

**Note:** Since \( f(x) \in B \) for every \( x \in A \), we don't separately define \( \text{PreIm}_f(B) \) — this is always just \( A \).

**Ex:** \( \Box \) \( A = \{1, 2, 3\} \quad f = \{(1, 3), (2, 0), (3, 0)\} \quad B = \{\ast, 0, 0\} \)
Then: \( \text{PreImp}(\emptyset) = \{x \in A \mid f(x) \in \emptyset \} \)
= \{x \in A \mid f(x) = \emptyset \}
= \emptyset.

\( \text{PreImp}(\{0, 13\}) = \{x \in A \mid f(x) \in \{0, 13\} \} \)
= \{x \in A \mid f(x) = 0, 13 \}
= \emptyset.

\( \text{PreImp}(\{0\}) = \{x \in A \mid f(x) \in \{0\} \} \)
= \{x \in A \mid f(x) = 0 \}
= \emptyset.

Consider \( f: \mathbb{R} \to \mathbb{R} \)
\( f(x) = x^2 \)

Then: \( \text{PreImp}(\{0, 13\}) = \{x \in \mathbb{R} \mid f(x) \in \{0, 13\} \} \)
= \{x \in \mathbb{R} \mid x^2 \in \{0, 13\} \}
= \{x \in \mathbb{R} \mid x^2 = 0, 13 \}
= \{x \in \mathbb{R} \mid x = 0, \sqrt{13}, -\sqrt{13} \}.

Also: \( \text{PreImp}(\{0, 27\}) = \{x \in \mathbb{R} \mid x^2 \in \{0, 27\} \} \)
= \{x \in \mathbb{R} \mid 0 \leq x^2 \leq 27 \}
= \{x \in \mathbb{R} \mid x^2 \leq 27 \}
= \{x \in \mathbb{R} \mid 0 \leq x \leq \sqrt{27} \} \)
\[ (\text{iii}) \]
\[ = [-\sqrt{2}, \sqrt{2}] \]

Also: \( \text{PreIm} \left( [0, \infty) \right) = \{ x \in \mathbb{R} | x^2 \in [0, \infty) \} = \mathbb{R} \).

Q: What happens if we take the preimage of the image of some \( x \in A \)?
or the image of the preimage of some \( y \in B \)?

Propn. Suppose \( f : A \to B \) is a function.

(i) Fix \( x \in A \)
    Then: \( \text{PreIm} \left( \text{Im} (x) \right) \supseteq x \)

(ii) Fix \( y \in B \)
    Then: \( \text{Im} \left( \text{PreIm} (y) \right) \subseteq y \)

Pf: (i) Fix \( x \in X \).
    By def'n: \( \text{PreIm} \left( \text{Im} (x) \right) = \{ y \in A | f(y) \in \text{Im} (x) \} \)
    but since \( x \in X \), we knew \( f(x) \in \text{Im} (x) \) by def'n of \( \text{Im} (x) \)
    Hence \( x \in \text{PreIm} \left( \text{Im} (x) \right) \)
    Since \( x \) was arbitrary, (i) is proved. \( \checkmark \)
(xiv) Fix \( y \in \text{Im}(\text{preIm}(Y)) \)
by def'n of image, \( \exists x \in \text{preIm}(Y) \)
s.t. \( f(x) = y \).
But then, by def'n of preimage, \( f(x) \in Y \), i.e. \( y \in Y \).
Since \( y \) was arbitrary, (iii) is proved.

Picture:

(i) \( \text{preIm}(f(x)) \)

(ii) \( \text{Im}(Y) \)

Note: in general neither containment can be reversed.

Ex: Let \( f: \mathbb{R} \to \mathbb{R} \)
\( \exists \ f(x) = x^2 \)
Let \( x = 2 \sqrt{3} \)
Then: \( \text{Imp}(x) = \text{Imp}(\{1\}) \)
\[ = \{ f(1) \} \]
\[ = \{ 1 \} = \{ 1 \}. \]

So: \( \text{PreImp}(\text{Imp}(x)) = \text{PreImp}(\{1\}) \)
\[ = \{ x \in \mathbb{R} \mid f(x) \in \{1\} \} \]
\[ = \{ x \in \mathbb{R} \mid x^2 \in \{1\} \} \]
\[ = \{ -1, 1 \}. \]

hence: \( x = \{1\} \Rightarrow \{ -1, 1 \} = \text{PreImp}(x) \)
in this case. \( \checkmark \)

New let \( y = \{ 2, -1 \} \)

Then: \( \text{PreImp}(y) = \{ x \in \mathbb{R} \mid f(x) \in \{2, -1\} \} \)
\[ = \{ x \in \mathbb{R} \mid x^2 \in \{2, -1\} \} \]
\[ = \{ -1, 1 \}. \]

So: \( \text{Imp}(\text{PreImp}(y)) = \text{Imp}(\{ -1, 1 \}) \)
\[ = \{ f(-1), f(1) \} \]
\[ = \{ (-1)^2, 1^2 \} \]
\[ = \{ 1 \}. \]

So \( \text{Imp}(\text{PreImp}(y)) = \{1\} \Rightarrow \{ 2, -1 \} = y \)
in this case. \( \checkmark \)
Let \( A = \{1, 2, 3\} \)
\( B = \{\ast, 0\} \)
\( C = \{1, 2\} \)
\( D = \{\ast, 0, D\} \)

Define
\[
\begin{align*}
g & : A \to B \\
h & : C \to D \\
j & : A \to D
\end{align*}
\]

by:
\[
\begin{align*}
g & = \{(1, \ast), (2, 0), (3, \ast)\} \\
h & = \{(1, \ast), (2, 0), (3, D)\} \\
j & = \{(1, \ast), (2, 0), (3, D)\}
\end{align*}
\]

Surjectivity: Def \( f : A \to B \)

is surjective (or onto) iff \( \text{Im} f = B \).

i.e. iff
\[
(\forall b \in B) \ (\exists a \in A) \ (f(a) = b)
\]
(xvii) \( \exists i, j \) above are surjective

- \( n \) is not; because \( D \notin \text{Im}_n \)

Proving surjectivity

**Example 1** Define \( f: \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \) by \( f(mn) = mn \).

Claim \( f \) is surjective

**Proof:** \( \text{WTS: } (\forall x \in \mathbb{Z}) (\exists (mn) \in \mathbb{Z} \times \mathbb{Z}) (f(mn) = x) \)

- So fix \( x \in \mathbb{Z} \)
- Observe \( f(0, x) = 0 + x = x \)
- Hence \( \exists (mn) \in \mathbb{Z} \times \mathbb{Z} \) s.t. \( f(mn) = x \), namely \( (m, n) = (0, x) \).
- Since \( x \) was arbitrary, claim is proved.

**Example 2** Define \( f: \mathbb{R} \to \mathbb{R} \) by \( f(x) = 2x + 1 \).

Claim: \( f \) is surjective

**Proof:** Fix \( y \in \mathbb{R} \).

- Let \( x = \frac{y - 1}{2} \)
- Then: \( f(x) = f\left(\frac{y - 1}{2}\right) = 2\left(\frac{y - 1}{2}\right) + 1 = y \)
- Since \( y \) was arbitrary, claim is proved.
(xviii) ③ Define \( f: \mathbb{R} \rightarrow \mathbb{R} \) by \( f(x) = x^2 \).

Claim: \( f \) is not surjective.

**Proof:**

\[
\text{wts: } \forall y \in \mathbb{R} \exists x \in \mathbb{R} (f(x) = y)
\]

\[
i.e. \ (\forall y \in \mathbb{R}) (\exists x \in \mathbb{R}) (f(x) = y)
\]

\[\iff y = -1.\]

\( f(x) = x^2 \geq 0 \)

\[\text{hence } f(x) \neq -1 = y.\]

Since \( x \) was arbitrary:

\[
(\forall x \in \mathbb{R}) f(x) \neq -1. \checkmark
\]

\((i.e. \ -1 \notin \text{Im} f)\).

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**Injective**

A function \( f: A \rightarrow B \) is called injective (or one-to-one or \((-1\)) if \( (\forall x, y \in A) (f(x) = f(y) \Rightarrow x = y) \)

or equivalently: \( (\forall x, y \in A) (x \neq y \Rightarrow f(x) \neq f(y)) \).

"Distinct inputs map to distinct outputs."

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**Ex.**

- \( g \) above is \( \underline{\text{not injective}} \) since \( 1 \neq 3 \)
  - \( g(1) = g(3) = * \)
- \( h, j \) are injective.
(2) Proving Injectivity:

Two approaches: fix \( x, y \in A \) and either:

1. Assume \( f(x) = f(y) \) and prove \( x = y \)
2. Assume \( x \neq y \) and prove \( f(x) \neq f(y) \).

Ex's 2 Consider again \( f: \mathbb{R} \rightarrow \mathbb{R} \) defined by \( f(x) = 2x + 1 \).

Claim: \( f \) is injective.

Proof:
- Fix \( x, y \in \mathbb{R} \)
- Assume \( f(x) = f(y) \)
- I.e. \( 2x + 1 = 2y + 1 \)
- Then \( 2x = 2y \) hence \( x = y \)
- Since \( x, y \) were arbitrary, claim is proved.

2. Define \( F: \mathbb{N} \rightarrow \mathbb{N} \) by \( F(n) = n^2 \).

Claim: \( F \) is injective.

Proof: Fix \( n, m \in \mathbb{N} \) and suppose \( n \neq m \).

WTS: \( F(n) \neq F(m) \).

Two cases:
1. \( n < m \)
2. \( m < n \)
(2c) if $\circ$ since $n \cdot m$ both positive can square both sides of inequality to get $n^2 < m^2$ i.e. $f(n) < f(m)$ so in particular $f(n) \neq f(m)$.

if $\bigcirc$ similar

since $n \cdot m$ were arbitrary, claim is proved.

3 Define $F : \mathbb{Z} \to \mathbb{Z}$ by $F(n) = n^2$

claim: $F$ is not injective

pf: $F(-2) = F(2) = 4$

but $-2 \neq 2$.

Bijections Def'n: a function $f : A \to B$ is bijective if $f$ is both injective and surjective

Ex: $g$ above is not bijective (surjective, but not injective)

- nor is $h$ (injective, but not surjective)

- $f$ is bijective
(xxi) Proving bijectivity:

Ex's

1. Consider again $F : \mathbb{R} \to \mathbb{R}$ defined by $F(x) = 2x + 1$.

   Claim: $F$ is bijective.

   Pf: We've already showed $F$ is both surjective and injective.

2. A spicier one: define $F : \mathbb{Z} \to \mathbb{N}$ by

   $$f(n) = \begin{cases} 2n & \text{if } n > 0 \\ 2(\lceil n \rceil) + 1 & \text{if } n \leq 0 \end{cases}$$

Picturing:

```
-3 -2 -1  0  1  2  3  4
   \_\_\_\_\_\_\_\_\_\_\_\_\_
   \_\_\_\_\_\_\_\_\_\_\_\_\_
   \_\_\_\_\_\_\_\_\_\_\_\_\_
```

Claim: $f$ is bijective.

Pf: (Surjectivity):

- Fix $n \in \mathbb{N}$
- If $n$ is even, then $n = 2k$ for some $k \in \mathbb{N}$ (so $k > 0$)
- Hence $f(k) = 2k = n$
If $n$ is odd, then $n = 2k + 1$ for some $k \in \mathbb{N}$; hence $k > 0$, hence $-k < 0$.

Hence $f(-k) = 2k + 1 = n$.

In either case, $(\exists x \in \mathbb{Z})(f(x) = n)$

Hence $f$ is surjective.

**Injectivity**

For $n, m \in \mathbb{Z}$ and assume $n \neq m$.

We wish $f(n) \neq f(m)$.

We may assume $n < m$, since case when $m < n$ is similar.

**Case 1**: $0 < n < m$.

Then $f(n) = 2n < 2m = f(m)$.

Hence $f(n) \neq f(m)$.

**Case 2**: $n < m \leq 0$.

Then $f(n) = 2(-n) + 1$

Then $f(m) = 2(-m) + 1$

**Observe**: Since $n < m$

$\Rightarrow$ $-n > -m$  
$\Rightarrow$ $2(-n) + 1 > 2(-m) + 1$  
$\Rightarrow$ $f(n) > f(m)$

So that $f(n) \neq f(m)$.

**Case 3**: $n \leq 0 < m$.

Then $f(n) = 2(-n) + 1$ is odd

Then $f(m) = 2m$ is even
(22iii)

Hence \( f(n) \neq f(m) \) in this case as well.

\( \Rightarrow \) Hence in all cases \( f(n) \neq f(m) \)

\( \Rightarrow \) Since \( n, m \) were arbitrary, we've proved 
\( f \) is injective.

Hence \( f \) is bijective. \( \checkmark \)