Homework #3

1. Prove or disprove each of the following statements:

   \[\bigcup_{n \in \mathbb{N}} P([n]) \subseteq P(\mathbb{N})\]

   \[P(\mathbb{N}) \subseteq \bigcup_{n \in \mathbb{N}} P([n])\]

2. For this problem, you may use the fact that \(\frac{1}{n}\) gets arbitrarily close to 0 as \(n\) gets larger and larger. That is, you may assume that for every \(z \in \mathbb{R}\) with \(z > 0\), there is an \(n \in \mathbb{N}\) such that \(\frac{1}{n} < z\).

   For each \(n \in \mathbb{N}\), define the sets \(A_n\) and \(B_n\) as follows:

   \[A_n = \left\{ x \in \mathbb{R} \mid 0 \leq x \leq \frac{n-1}{n} \right\}\]

   \[B_n = \left\{ y \in \mathbb{R} \mid -\frac{1}{n} < y < 1 \right\}\]

   Use a double containment argument to prove that

   \[\bigcup_{n \in \mathbb{N}} A_n = \bigcap_{n \in \mathbb{N}} B_n\]

3. Consider the following proposition \(P\):

   \[(\forall x \in \mathbb{R})(\exists y \in \mathbb{R})(x^2 - y^2 \geq 0)\]

   Write \(\neg P\) in positive form, that is, write down a statement logically equivalent to \(\neg P\) with the negation symbol inside the quantifiers (or, better yet, with no negation symbol). Then determine if \(P\) or \(\neg P\) is true. If \(P\) is true, prove it. If \(\neg P\) is true, then prove \(\neg P\).

4. Write out the following statements symbolically in positive form and determine whether they are true or false (no proof required).

   - There is no real number whose square is \(-1\).
   - If an integer \(n\) has a multiplicative inverse in the integers, then \(n\) must be 0 or 1.
   - For any real numbers \(x\) and \(y\), if \(x\) and \(y\) are both nonpositive then their product is nonnegative.
   - The product of two odd integers is not even.