Function

- Functions, like relations, are ubiquitous in math.
- But what "are" functions?
- Intuitively: a rule that assigns to each \( x \) in a domain \( A \) a unique output \( f(x) \) in a codomain \( B \).

\[ \Rightarrow \text{Can define functions rigorously as a special type of relation} \]

Definition: A function (with domain \( A \) and codomain \( B \)) is a relation \( f \subseteq A \times B \) such that for every \( a \in A \) there is a unique \( b \in B \) such that \((a, b) \in f\).

\[ (\forall a \in A) \exists! b \in B [(a, b) \in f \land (\forall c \in B) ((a, c) \in f \Rightarrow b = c)] \]

\[ \Rightarrow \text{we write } f : A \rightarrow B \]

To indicate that a subset \( f \subseteq A \times B \) is a function.

\[ \Rightarrow \text{we also write } f(a) = b \]

To mean \((a, b) \in f\).
Note: - the definition says every \( a \in A \) is assigned on output \( f(a) \in B \)
- does not insist that for every \( b \in B \) there is \( a \in A \) s.t. \( f(a) = b \)
(fraction \( u \) has property one called onto)

**Example:**

Let \( A = \{1, 2, 3\} \) \( B = \{\ast, 0, 43\} \)

Then \( f = \{(1, \ast), (2, 0), (3, \ast)\} \) is a function from \( A \) to \( B \)

```
A   B
\( 1 \rightarrow \ast \)
\( 2 \rightarrow 0 \)
\( 3 \rightarrow \ast \)
```

but \( g = \{(1, \ast), (1, 0), (2, \ast), (3, \ast)\} \) is not a function since \( 1 \) does not have a unique output

```
A   B
\( 1 \rightarrow \) not a function
\( 2 \rightarrow \ast \)
\( 3 \rightarrow \ast \)
```

Nor \( u = \{(2, \ast), (3, 0)\} \) since \( 1 \) is not assigned on output
We'll often define functions by some rule, e.g.

\[ F: \mathbb{R} \to \mathbb{R} \]
\[ F(x) = x^2 \]

or

\[ F: \mathbb{R} \to \mathbb{Z} \]
\[ F(x) = \lfloor x \rfloor \]

but behind the scenes we still consider these \( F \)'s to be sets of ordered pairs.

E.g. if we define \( F(x) = x^2 \),
then \((2, 4) \in F\)
\((3, 9) \in F\)
\((4, 16) \notin F\)

Warning: not all rules yield well-defined functions.

E.g. suppose we "define" by the rule
\[ F(\text{m,n}) = \text{m+n} \]
then this "function" is not one

\[ f(Y_2) = 1 + 2 = 3 ≠ 6 = 2 + 4 = f(2/4) \]

but \( Y_2 = 2/4 \).

- so \( f \) assigns multiple outputs to the same input.
- What's going on? really there's an implicit equiv. relation on \( F \).
  
  \[ \frac{1}{2} = \frac{2}{4} = \frac{3}{6} = \ldots \]

- our rule defines \( f \) on a representative of an equiv. class

- in general: when given a rule "defining" some \( F \subseteq A \times B \) to verify \( F \) is a function one must show:
  
  1. \( \forall a \in A \exists b \in B \) s.t. \((a,b) \in F\)
  2. if \( a = a' \) then \( f(a) = f(a') \)

**Equality of Functions**

Q: what does it mean for functions \( F : A \rightarrow B \) and \( g : A \rightarrow B \) to be equal?

A: \( F = g \) iff they're equal as sets of ordered pairs, i.e. \( F \subseteq g \).

- equality, \( \iff \) \( f = g \) iff \((a,b) \in F \iff (a,b) \in g\).
In practice, easier to see following:

**Theorem:** If $F: A \to B$ and $G: A \to B$ are functions then $F = G$ iff $(\forall a \in A) (F(a) = G(a))$.

**Pf:** you try.

The point: functions can be equal despite being defined by different rules.

**ex:** let $A = \{1, 2, 3\}$

Define $F: A \to N$ and $G: A \to N$

by

\[
F(x) = x^3 + 11x \\
G(x) = 6x^2 + 6
\]

Then

\[
F(1) = 12 = G(1) \\
F(2) = 30 = G(2) \\
F(3) = 60 = G(3)
\]

i.e. $F = \{(1, 12), (2, 30), (3, 60)\} = G$.

(What's the magic trick?)

\[
F - G = x^3 - 6x^2 + 11x - 6 = (x-1)(x-2)(x-3)
\]
Image

Def'n. Suppose \( f : A \rightarrow B \) is a function and \( X \subseteq A \).

The image of \( X \) under \( f \), denoted \( \text{Imp}(X) \), is defined as:

\[
\text{Imp}(X) = \{ b \in B \mid \exists x \in X \text{ such that } f(x) = b \}
\]

more informally we write:

\[
\{ f(a) \mid a \in X \}
\]

when \( X = A \) we just say that \( \text{Imp}(A) \) is the image of \( f \) and sometimes just write \( \text{Imp} \)

Def'n says: \( \text{Imp}(X) \) is the "set of outputs \( f(x) \) in \( X \)"

\( \text{Imp} = \text{Imp}(A) \) is the "set of all outputs."

In particular:

\[
\text{if } x \in X \text{ then } f(x) \in \text{Imp}(X)
\]

Picture:

[Diagram of a function mapping from set A to set B with \( \text{Imp}(A) = \text{Imp} \) and \( \text{Imp}(x) \).]
Ex: ① \( \text{Let } A = \{1, 2, 3\} \)
\( B = \{*, 0, A\} \)
\( f = \{(1, *), (2, 0), (3, *)\} \)

Then: \(-\text{Imp}(E_{1, 3}) = \{F(1), F(3)\}\)
\[= \{*, *\}\]
\[= \{*\}\]

\(-\text{Imp} = \text{Imp}(A) = \{F(1), F(2), F(3)\}\)
\[= \{*, 0, *\}\]
\[= \{*, 0, *\}\]

② \( \text{Let } f: \mathbb{R} \rightarrow \mathbb{R} \text{ be defined by } f(x) = x^2 \)

Then: \(-\text{Imp}([1, 0, 1])\)
\[= \{(-1)^2, 0^2, 1^2\}\]
\[= \{1, 0, 1\}\]

\(-\text{Imp} = \{x \in \mathbb{R} \mid x \geq 0\}\)

Functions add a layer of complexity to basic set theory of \(\mathbb{N}, \mathbb{U}, \ldots\) we studied earlier.

Prop: Suppose \( F: A \rightarrow B \) is a function and \( S, T \subseteq A \).

Then:
\(-\text{Imp}(S \cap T) \leq \text{Imp}(S) \cap \text{Imp}(T)\)
PF:-fix y \in \text{Imf}(\text{sat})
-\text{then } \exists x \in \text{sat} \text{ s.t. } f(x) = y
-\text{hence } x \in S \text{ and } x \in T
-\text{hence } f(x) \in \text{Imf}(S) \text{ and } f(x) \in \text{Imf}(T)
-\text{i.e. } y \in \text{Imf}(S) \text{ and } y \in \text{Imf}(T)
-\text{i.e. } y \in \text{Imf}(S) \cap \text{Imf}(T)

Since y was arbitrary the prop'n u proved.

Note: in general we don't have
\text{Imf}(\text{sat}) = \text{Imf}(S) \cap \text{Imf}(T)

e.g. Consider \( f(x) = x^2 \) on \( \mathbb{R} \).
let \( S = [-1,0] \), \( T = [0,1,2,3] \)
then:
\text{Imf}(S) = \{ f(-1), f(0) \}
= \{ 1, 0 \}
\text{Imf}(T) = \{ f(0), f(1), f(2) \}
= \{ 0, 1, 4 \}
\text{hence } \text{Imf}(S) \cap \text{Imf}(T) = \{ 0, 1 \}

but:
\text{Imf}(\text{sat}) = \text{Imf}(\{ 0 \})
= \{ f(0) \}
= \{ 0 \}

So in this case
\text{Imf}(\text{sat}) \not\subseteq \text{Imf}(S) \cap \text{Imf}(T).

\text{The essence: Anarchists can send multipr
inputs to some output!}
### Preimages

**Definition:** Suppose $f : A \rightarrow B$ is a function and $Y \subseteq B$. The **preimage** of $Y$ under $f$, denoted $\text{PreImp}(Y)$, is defined as:

$$\text{PreImp}(Y) = \{ x \in A \mid f(x) \in Y \}$$

**Note:** Since $f(x) \in B$ for every $x \in A$, we don't separately define $\text{PreImp}(B)$, since this is always just $A$. 

The diagram illustrates the relationship between $\text{Imp}(t)$, $\text{Imp}(s)$, and $\text{Imp}(s \circ t)$, showing how elements are mapped between sets.
ex. 1) \[ A = \{1, 2, 3\} \]
\[ B = \{\ast, 0, \emptyset\} \]
\[ F = \{(1, \ast), (2, 0), (3, \ast)\} \]

Then: \[ \text{PreImf}(\{\ast\}) = \{x \in A \mid f(x) \in \{\ast\}\} \]
\[ = \{x \in A \mid f(x) = \ast\} \]
\[ = \{1, 3\} \]

\[ \text{PreImf}(\{\ast, 0\}) = \{x \in A \mid f(x) \in \{\ast, 0\}\} \]
\[ = \{1, 2, 3\} = A \]

\[ \text{PreImf}(\emptyset) = \{x \in A \mid f(x) \in \emptyset\} \]
\[ = \emptyset \]

(2) Let \( f: \mathbb{R} \rightarrow \mathbb{R} \) be defined by \( f(x) = x^2 \)

Then \[ \text{PreImf}(\{0, 1\}) \]
\[ = \{x \in \mathbb{R} \mid f(x) \in \{0, 1\}\} \]
\[ = \{x \in \mathbb{R} \mid x^2 \in \{0, 1\}\} \]
\[ = \{-1, 0, 1\} \]

\[ \text{PreImf}(\{0, 2\}) \]
\[ = \{x \in \mathbb{R} \mid x^2 \in \{0, 2\}\} \]
\[ = \{x \in \mathbb{R} \mid 0 \leq x^2 \leq 2\} \]
\[ = \{x \in \mathbb{R} \mid -\sqrt{2} \leq x \leq \sqrt{2}\} \]
\[ = [-\sqrt{2}, \sqrt{2}] \]
\[ \text{PreImp}(\text{co}x_0) = \{ x \in \mathbb{R} \mid x^2 \in \text{co}x_0 \} = \mathbb{R}. \]

\[ \text{PreImp}(\mathbb{R}) = \mathbb{R}. \]

Q: What if we take the preimage of the image of \( x \in A \)?
or the image of the preimage of \( y \in B \)?

Proof:
Suppose \( f: A \rightarrow B \) is a function.

(i) Fix \( x \in A \)
Then \[ \text{PreImp}(\text{Imp}(x)) = x \]

(ii) Fix \( y \in B \)
Then \[ \text{Imp}(\text{PreImp}(y)) \leq y \]

Proof: (i) Fix \( x \in X \).

By def'n: \[ \text{PreImp}(\text{Imp}(x)) = \{ y \in A \mid f(y) \in \text{Imp}(x) \} \]

Now, we know \( f(x) \in \text{Imp}(x) \) by def'n of \( \text{Imp}(x) \)

Hence \( x \in \text{PreImp}(\text{Imp}(x)) \)
Since \( x \) was arbitrary, (i) is proved.
(ii) Fix $y \in \text{Imp}(\text{PreImp}(Y))$
then $\exists z \in \text{PreImp}(Y)$ s.t. $f(z) = y$

but by definition $\text{PreImp}(Y) = \{ x \in A | f(x) \in Y \}$

hence $f(z) \in Y$

hence $y \in Y$

since $y$ was arbitrary

(iii) is proved

Pitches:

(i)

(ii)
In general, neither containment can be reversed.

(i) e.g. \( f : \mathbb{R} \rightarrow \mathbb{R} \)
\[ f(x) = x^2 \]
\[ \text{Let } x = \{1\} \]

Then: \( \text{Imp}(x) = \text{Imp}(\{1\}) = \{ f(x) \} = \{1\} \)

\[ \Rightarrow \text{PreImp}(\text{Imp}(x)) = \text{PreImp}(\{1\}) = \{ x \in \mathbb{R} \mid x^2 = 1 \} = \{-1, 1\} \neq x \]

Hence, \( X \neq \text{PreImp}(\text{Imp}(x)) \), in this case.

(ii) \( \text{Let } y = \{-5, 13\} \)
Then: \( \text{PreImp}(y) \)
\[ = \{ x \in \mathbb{R} \mid f(x) \in \{-5, 13\} \} \]
\[ = \{ x \in \mathbb{R} \mid x^2 \in \{-5, 13\} \} \]
\[ = \{-1, 1\} \]

Hence, \( \text{Imp}(\text{PreImp}(y)) \)
\[ = \text{Imp}(\{-1, 1\}) \]
\[ = \text{Pre} \{ \text{f}(-1), f(1) \} = \{1\} \neq y \].
Injections

Let $A = \{1, 2, 3\}$
$B = \{\ast, 0\}$
$C = \{1, 2\}$
$D = \{\ast, 0, \Delta\}$

Define:
- $g: A \rightarrow B$
- $h: C \rightarrow D$
- $j: A \rightarrow D$

By:
- $g = \{(1, \ast), (2, 0), (3, \ast)\}$
- $h = \{(1, \ast), (2, 0), (3, \ast)\}$
- $j = \{(1, \ast), (2, 0), (3, \Delta)\}$

Surjective:

Definition: a function $f: A \rightarrow B$ is surjective (or onto) iff $\text{Im}(f) = B$.

Hence, iff

$(\forall b \in B) \ (\exists a \in A) \ (f(a) = b)$
(ii)

- \( x^5 = g \) and \( j \) above are surjective. \( h \) is not because \( y^3 \).

Proving surjectivity

\[ \text{ex: } 1 \text{ Define } F: \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \]
\[ \text{by } F((m,n)) = m+n. \]

Claim: \( F \) is surjective

PF: \( \forall x \in \mathbb{Z} \) \((\exists (m,n) \in \mathbb{Z} \times \mathbb{Z}) F((m,n)) = x \)

- So fix \( x \in \mathbb{Z} \)
- observe \( F((0, x)) = 0 + x = x \)
- Hence \( \exists (m,n) \) s.t. \( F((m,n)) = x \)
  - Namely \( (m,n) = (0, x) \),
- Since \( x \) was arbitrary, claim is proved.

2) Define \( F: \mathbb{R} \to \mathbb{R} \) by
\[ f(x) = 2x + 1 \]

Claim \( f \) is surjective.

PF: - Fix \( y \in \mathbb{R} \)
  - \( \exists x \) s.t. \( x = \frac{y-1}{2} \)
  - Then \( f(x) = 2\left(\frac{y-1}{2}\right) + 1 = y-1 + 1 = y \)
- Since \( y \) was arbitrary, claim is proved.
Define $f: \mathbb{R} \to \mathbb{R}$ by $f(x) = x^2$.

Claim: $f$ is not surjective.

Proof: WTS: $\forall y \in \mathbb{R}, \exists x \in \mathbb{R}$ such that $f(x) = y$.

Counterexample: $y = -1$.

Then $\forall x \in \mathbb{R}$, $f(x) = x^2 \geq 0$.

Thus $f(x) \neq -1$.

Injective

Definition: A function $f: A \to B$ is called injective (or one-to-one or 1-1) if

$$(\forall x, y \in A)(f(x) = f(y) \Rightarrow x = y)$$

Sometimes helpful to write definition in contrapositive form:

$$(\forall x, y \in A)(x \neq y \Rightarrow f(x) \neq f(y))$$

"Distinct inputs map to distinct outputs."

Example: $g$ above is not injective.

Since $1 \neq 3$ but $g(1) = g(3) = a$.

Thus $h, j$ are injective.
Proving injectivity

Two approaches: Fix \( x, y \in A \) and another:

1. Assume \( f(x) = f(y) \), prove \( x = y \)
2. Assume \( x \neq y \), prove \( f(x) \neq f(y) \)

Ex's

1. Define \( f: \mathbb{R} \to \mathbb{R} \) by \( f(x) = 5x + 6 \)

Claim: \( f \) is injective

PF: Fix \( x, y \in \mathbb{R} \)
- Assume \( f(x) = f(y) \)
  \[ 5x + 6 = 5y + 6 \]
  \[-\text{Subtract } 6 \quad \text{from both sides} \]
  \[ 5x = 5y \]
  \[-\text{Divide } 5 \quad \text{on both sides} \]
  \[ x = y \]

Since \( x, y \) were arbitrary, claim is proved.

2. Define \( f: \mathbb{N} \to \mathbb{N} \) by \( f(n) = n^2 \)

Claim: \( f \) is injective

PF: Fix \( n, m \in \mathbb{N} \) and assume \( n \neq m \)
- (WTS: \( f(n) \neq f(m) \))
  - Two cases: (i) \( n < m \)
    - (ii) \( m < n \)

If (i): Since \( n, m \) both positive we may square both sides...
\[ c^2 < \frac{1}{2} \]

i.e. \( f(n) < f(m) \)

hence \( f(n) \neq f(m) \)

(ii) Similar

\( \Rightarrow \) since \( n \neq m \) we arbitrary claim is proved

\( \mathcal{C} \) Define \( f : \mathbb{Z} \rightarrow \mathbb{Z} \) \( f(n) = n^2 \)

Claim: \( f \) is not injective

PF: \( f(-2) = f(2) = 4 \)

- but \( -2 \neq 2 \)

- hence \( f \) is not injective.

Bijections

Def'n a function \( f : A \rightarrow B \) is \underline{bijective} iff it is both \underline{injective} and \underline{surjective}

ex: \( g \) is not a bijection

- \( g \) is surjective, but \( \text{not injective} \)

- \( h \) is \( \text{injective} \) but \( \text{not surjective} \)

\( j = \) \underline{bijective}

i.e. \( j \) is a bijection
(vi)

Proving bijection

1. Define \( f: \mathbb{R} \rightarrow \mathbb{R} \) by
   \[ f(x) = 3x - 1 \]

Claim: \( f \) is a bijection (i.e. surjective and injective)

Proof (Surjectivity): Fix \( y \in \mathbb{R} \)

Let \( x = \frac{y + 1}{3} \)

Then \( f(x) = 3\left(\frac{y + 1}{3}\right) - 1 \)

\[ = y + 1 - 1 \]

\[ = y \]

Since \( y \) was arbitrary, \( f \) is surjective.

Proof (Injectivity): Fix \( x, y \in \mathbb{R} \) assume

\( f(x) = f(y) \)

i.e. \( 3x - 1 = 3y - 1 \)

Then \( 3x = 3y \) i.e. \( x = y \)

Since \( x, y \) arbitrary, \( f \) is injective.

Hence \( f \) is bijective as claimed.

2. Define \( F: \mathbb{Z} \rightarrow \mathbb{N} \) by:

\[ f(n) = \begin{cases} 2n & \text{if } n > 0 \\ 2(n-1) + 1 & \text{if } n \leq 0 \end{cases} \]
Claim: \( f \) is a bijection

**PF: (Surjectivity):**
- Fix \( n \in \mathbb{N} \)
  - (If \( n \) is even, then \( n = 2k \) for some \( k \in \mathbb{N} \) (hence \( k > 0 \))
  - Hence \( f(k) = 2k = n \)
  - If \( n \) is odd, then \( n = 2k + 1 \) for some \( k \in \mathbb{N} \) (hence \( k > 0 \))
  - Hence \( f(-k) = 2k + 1 = n \)

Thus, in either case \( \exists k \in \mathbb{N} \) \( f(k) = n \)
- Hence \( f \) is surjective.

**Injectivity:**
- Fix \( n, m \in \mathbb{Z} \) and assume \( n \neq m \)
- We may assume \( n < m \).
- Since \( f \) is a bijection, the argument is similar.
Case 1: \( 0 < n < m \)
- Then \( F(n) = 2n < 2m = F(m) \)
- Hence \( F(n) \neq F(m) \)

Case 2: \( n < m \leq 0 \)
- Then \( F(n) = 2(-n) + 1 \)
  \( F(m) = 2(-m) + 1 \)
- Observe: since \( n < m \)
  \( \Rightarrow -n > -m \)
  \( \Rightarrow 2(-n) > 2(-m) \)
  \( \Rightarrow 2(-n) + 1 > 2(-m) + 1 \)
  (i.e. \( F(n) > F(m) \))
- Hence \( F(n) \neq F(m) \) in this case as well.

Case 3: \( n \leq 0 < m \)
- Then \( F(n) = 2(-n) + 1 \) is odd
  \( F(m) = 2m \) is even
- Hence \( F(n) \neq F(m) \) in this case as well.

\( \Rightarrow \) Hence in all cases \( F(n) \neq F(m) \)
\( \Rightarrow \) Since \( n, m \) arbitrary \( F \) is injective
\( \Rightarrow \) Hence \( F \) is bijective
Compositions

Def'n Sps \( F : A \to B \) and \( g : B \to C \) are functions.

The \textit{composition} \( g \circ f \) of \( f \) and \( g \) directed \( g \circ f \), is defined by, \( \forall x \in A \)

\[ g \circ f (x) = g(f(x)) \]

\[\begin{array}{ccc}
  & & g \circ f (x) \\
  \downarrow & & \downarrow \\
  \circ f (x) & \rightarrow & g(f(x)) \\
  x & \rightarrow & f(x) \\
  & & C
\end{array}\]

\[\begin{array}{ccc}
  A & \rightarrow & B \\
  & & \downarrow \circ f \\
  & & f(x) \\
  & & C \\
  x & \rightarrow & B
\end{array}\]

\textbf{Ex.}: Define \( F : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \)

by \( f(m,n) = m+n \)

\( g : \mathbb{Z} \to \mathbb{N} \)

by \( g(n) = n^2 + 1 \)

Then \( g \circ f (1,3) = g(f(1,3)) \)

\[ = g(4) \]

\[ = 17 \]

In general: \( g \circ f (m,n) \)

\[ = g(f(m,n)) \]

\[ = g(m+n) \]

\[ = (m+n)^2 + 1 \]
The Identity Function

Def'n: Let $A$ be a fixed set. The identity function on $A$, denoted $\mathrm{id}_A$, is the function defined by:

$$\mathrm{id}_A : A \to A$$

$$\forall x \in A \quad \mathrm{id}_A(x) = x.$$ 

E.g., if $A = \{\#, 0, \oslash, \Delta\}$, then $\mathrm{id}_A : A \to A$ is:

$$\mathrm{id}_A = \{ (\#, \#), (0, 0), (\oslash, \oslash), (\Delta, \Delta) \}$$

Def'n: Let $f : A \to B$ be a function. Then $f$ is invertible if there exists a function $g : B \to A$ such that:

$$g \circ f = \mathrm{id}_A \quad \text{and} \quad f \circ g = \mathrm{id}_B.$$ 

$g$ is called the inverse of $f$, and $f$ is denoted $f^{-1}$.

Note: not all functions are invertible!
In fact:

**Theorem:** Let $F: A \to B$ be a function. Then $F$ is invertible if and only if $F$ is a bijection.

$(\Rightarrow)$ Suppose $F$ is invertible. Let $g$ be its inverse. We prove $F$ is a bijection.

**(surjectivity):** Fix $y \in B$.
- Let $x = g(y)$.
- Then $F(x) = F(g(y)) = F(g(y)) = y$.

Since $y$ is arbitrary, $F$ is surjective.

**(injectivity):** Fix $x, y \in A$ and suppose $F(x) = F(y)$.
- Then $g(F(x)) = g(F(y))$.
- $x = y$.

Since $x$ and $y$ are arbitrary, $F$ is injective.

Hence $F$ is a bijection.

$(\Leftarrow)$ Suppose $F$ is a bijection from $A$ to $B$. We prove $F$ is invertible.

Define $g = \{(b, a) \in B \times A \mid (a, b) \in F\}$.

We prove $g = F^{-1}$.

\[ g(b) = a \quad (a, b) \in F \Rightarrow (b, a) \in g \]
Claim 1: \( g \) is a function from \( B \) to \( A \).

**Proof:**

- **WTS:** \( \forall b \in B \exists a \in A \text{ s.t. } (b, a) \in g \).

  - **Existence:** Fix \( b \in B \). Since \( g \) is surjective, \( \exists a \in A \) s.t. \( f(a) = b \).
    - Hence, \( (b, a) \in g \).

  - **Uniqueness:** Suppose there is \( a' \in A \) s.t. \( (b, a') \in g \).
    - Then \( f(a') = b \) (by def'n of \( g \)) and \( (a, b) \in F \).
    - But then \( f(a') = f(a) \)
    - Hence, since \( f \) is surjective, \( a = a' \). \( \checkmark \)

Claim 2: \( g = f^{-1} \)

**Proof:**

- Fix \( a \in A \).
  - Let \( b = f(a) \), so that \( (a, b) \in F \).
  - Then \( (b, a) \in g \) (i.e., \( g(b) = a \)).
    - Hence \( g(f(a)) = g(b) = a \).
    - Since \( a \) arbitrary, \( g \circ f = 1_A. \) \( \checkmark \)

- Fix \( b \in B \).
  - Let \( a = g(b) \), i.e. \( (b, a) \in g \).
    - Then \( (a, b) \in F \) (by def'n of \( g \))
    - Hence \( f(g(b)) = f \circ g(b) = b \).
    - Since \( b \) arbitrary, \( f \circ g = 1_B. \) \( \checkmark \)

Here \( g \) is inverse of \( f. \) \( \checkmark \)
Can use theorem to prove certain functions are bijections.

**Ex:** Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = 2x - 5$.

**Claim** $f$ is a bijection.

**Proof:** We show $f$ is invertible.

- Let $g : \mathbb{R} \to \mathbb{R}$ be defined by $g(x) = \frac{x + 5}{2}$.
- We show $g = f^{-1}$.

**New:** Fix $x \in \mathbb{R}$.

- $g \circ f(x) = g(f(x)) = g(2x - 5) = \frac{2x - 5 + 5}{2} = x$.

Thus $g \circ f = 1_{\mathbb{R}}$.

Also, $f \circ g(x) = f(g(x)) = f\left(\frac{x + 5}{2}\right) = 2\left(\frac{x + 5}{2}\right) - 5 = x$.

Thus $f \circ g = 1_{\mathbb{R}}$.

Hence $f$ is invertible.

Hence $f$ is a bijection, by previous theorem.