Partitions yield equiv. relations

Idea: If P is a partition on A, can define equiv. relation R on A by rule "(x,y) ∈ R if f.x and y are in some piece of partition."

Picture

\[(x,y) ∈ R \text{ but } (x,z) \notin R\]

Let's prove this works:

**Theorem** Sps P is a partition of A.
Define a relation \( R_p \) on A by:

\[(x,y) ∈ R_p \text{ if } \exists x ∈ P \text{ s.t. } x ∈ X \text{ and } y ∈ X.\]

Then \( R_p \) is an equivalence relation.

**Pf:** (i) reflexivity:
- Fix \( x ∈ A \).
- Since IP is a partition of A, there is \( X ∈ P \) s.t. \( x ∈ X \).
- hence \( x \in x \) else.

- hence \( (x,x) \in R_p \)

(iii) **Symmetry**

- Fix \( x, y \in A \) and suppose \( (x, y) \in R_p \)
- Then there \( u \in E \) s.t. \( x \in x \) and \( y \in y \)
- hence \( y \in x \) and \( x \in x \)
- hence \( (y, x) \in R_p \)

(iii) **Transitivity**

- Fix \( x, y, z \in A \) and suppose \( (x, y) \in R_p \) and \( (y, z) \in R_p \)
- then \( \exists x \in E \) s.t. \( x \in x \) and \( y \in y \) and \( \exists y \in E \) s.t. \( y \in Y \) and \( z \in Y \)
- hence \( x \in x \) \( y \in Y \)
- in particular \( x \neq \emptyset \), so that \( X = Y \) (since \( R_p \) is a partition)
- hence \( x \in x \) and \( z \in z \)
- i.e. \( (x, z) \in R_p \)

**Ex.** (i) - Let \( P = \{ X_n : n \in \mathbb{Z} \} \) be our partition of \( P \) from last time, i.e. \( X_n = C \cup C_n + 1 \)

- Let \( R_p \) be the associated equivalence relation, \( (x, y) \in R_p \) iff

\[ \exists x \in x \] \( x \in x \) and \( y \in Y \),

i.e. \( x \in (C_n + 1) \) and \( y \in C \).

- by our theorem, this defines an equivalence relation.
- easy to see this is the
Some equiv. relation $R$ that we defined previously in a different way: $(x,y) \in R$ iff $|x| = |y|$

$\overline{x_n \ x_{n+1}}$

$(x,y) \in R \iff R_{\text{ip}}$

$(x,z) \notin R \iff R_{\text{ip}}$

Notice: the equivalence classes of this equiv. relation are exactly the pieces in the partition

(2) Let $P = \{ [1,3], [2,3,4,3] \}$ be our partition of $A = \{1,2,3,4\}$ from last time.

Let $R_{\text{ip}}$ be the associated equiv. relation.

So e.g. $(1,1) \in R_{\text{ip}}$

$(2,3) \in R_{\text{ip}}$

but $(1,2) \notin R_{\text{ip}}$

In this case we can explicitly write out $R_{\text{ip}}$ as a set in roster notation:

$R_{\text{ip}} = \{ (1,1), (2,2), (3,3), (4,4), (2,3), (3,2), (2,4), (4,2), (3,4), (4,3) \}$
- no real rhyme or reason to this equiv. relation, but still a perfectly good one.

Equiv. relations yield partitions

- Summary of above: given a partition \( P \) of \( A \), one can define an equiv. relation \( R_P \) by saying, the equivalence classes of \( R_P \) are exactly the pieces of the partition \( P \).

- Conversely: given an equiv. relation \( R \) on \( A \), (you will prove on your own) the equiv. classes of \( R \) always form a partition of \( A \).

**Defn**: Sps \( R \) is an equiv. relation on \( A \). We denote the set of equiv. classes of \( R \) as \( A/R \)

\[ A/R = \{ [x]_R : x \in A \} \]

"\( A \) mod \( R \)"

**Existence - Consider \( \equiv_3 \) on \( \mathbb{Z} \).**

Then:

\[ \mathbb{Z}/\equiv_3 = \{ [n]_3 : n \in \mathbb{Z} \} \]
\[ \{ \ldots, [-1]_3, [0]_3, [1]_3, [2]_3, \ldots \} \]

We checked already:

\[ (-1)_3 = [0]_3 = [3]_3 = [6]_3 \quad \cdots \]
\[ [1]_3 = [4]_3 = [7]_3 \quad \cdots \]
\[ [2]_3 = [5]_3 = [8]_3 \quad \cdots \]

So really:

"set of remainders" \( \mathbb{Z}/\equiv_3 = \{ [0]_3, [1]_3, [2]_3 \} \)

we could as well write:

\[ \mathbb{Z}/\equiv_3 = \{ [3]_3, [4]_3, [5]_3 \} \]

**NOTATION:** it is convenient to write \( \mathbb{Z}/\equiv_n \) as \( \mathbb{Z}/n\mathbb{Z} \)

"\( \mathbb{Z} \) mod \( n\mathbb{Z} \)"

just like with 3, in general we have:

\[ \mathbb{Z}/n\mathbb{Z} = \{ [0]_n, [1]_n, \ldots, [n-1]_n \} \]

\( \mathbb{Z}/n\mathbb{Z} \) is a "floor" equivalence relation on \( \mathbb{Z} \), \( \langle x, y \rangle \in \mathbb{Z}/n\mathbb{Z} \) if \( \lfloor x \rfloor = \lfloor y \rfloor \)
- we knew from before: equiv.
  classes are intervals of the
  form \([Ch, n+1)\)

- if \(x \in \text{Ch}, n+1)\) then \([xJ]_R = \text{Ch}, n+1)\)

- any \(x\) in this interval serves
  equally well as a representative
  of the equiv. class

- so e.g. \([0J]_R = [1/2J]_R = [3/8J]_R\)
  \([0, 1)\)

\([1J]_R = [1.2121\ldots]_R\)
  \([1.99_J]_R\)
  \([1, 2)\)

etc.

We have:

\[ R/R = \{ [xJ]_R : x \in R \} \]

- \([-1, 0, 1, 2, \ldots) = \{ \ldots, [-1, 0), [0, 1), [1, 2), \ldots \} \]

- \([-1J]_R, [0J]_R, [1J]_R \}

- \([-1/2J]_R, [1/2J]_R, [3/2]_R \}

etc.
In both ex's 0 and 2 the set of equiv classes forms a partition.

Thus if R is an equiv relation on A, then A/R is a partition of A.

**Proof:** Hw. For hint, see problem 6.7.13 pg. 449, which outlines the proof.

**Order Relations**

Neither common type of binary relation is an order relation. Come in several flavors:
- Reflexive/strict/total.
- Antisymmetric.
- Transitive.

**Definition:** A relation R on a set A is a (strict) partial order if R is reflexive, transitive, and antisymmetric.

If R is a partial order on A we say that the pair (A, R) is a partially ordered set or poset.
Exercise 1. Let $\leq$ be a partial order on $\mathbb{R}$.

Proof: Let $x, y, z \in \mathbb{R}$.
- If $x \leq y$ and $y \leq x$ then $x = y$.
- If $x \leq y$ and $y \leq z$ then $x \leq z$.

So $(\mathbb{R}, \leq)$ is a poset.

2. Let $A$ be any set. Then the subset relation $\subseteq$ on $\mathcal{P}(A)$ is a partial order.

Proof: Let $X, Y, Z \in \mathcal{P}(A)$.
- If $X \subseteq Y$ and $Y \subseteq X$ then $X = Y$.
- If $X \subseteq Y$ and $Y \subseteq Z$ then $X \subseteq Z$.

So $(\mathcal{P}(A), \subseteq)$ is a poset.

3. We showed before that the divisibility relation on $\mathbb{N}$ is reflexive, transitive, and antisymmetric, hence $(\mathbb{N}, |)$ is a poset.

We also showed that $(\mathbb{Z}, \leq)$ is not a poset.

These exercises seem to be of different kinds, yet any theorem...
that can be proved using only properties of reflexivity, transitivity and antisymmetry must hold for all three (and any other poset).

**Strict p.o.'s**

**Defn** a relation $R$ on $A$ is called **irreflexive** iff $(\forall x \in A) (x, x) \notin R$.

*E.g.* $<$ and $\neq$ are irreflexive since we never have $x < x$ or $x \neq x$.

**Defn** a relation $R$ on a set $A$ is called a **strict partial order** iff $R$ is irreflexive, transitive, and antisymmetric.

This is official def'n, but by how this is equiv to being transitive and *symmetric* $\forall y, z \in A$ $xy \rightarrow z$.

**Ex's** $<$ is a strict partial order on $R$.

*Proof*: $\forall x, y, z \in \mathbb{R}$ we have:

*Reflexive*: $x \neq x \checkmark$ 
*Symmetric*: $x < y \checkmark \leftarrow y < x$ 
*Transitive*: $x < y \wedge y < z \Rightarrow x < z \checkmark$ 
*Antisymmetric*: $x < y \wedge y < x \Rightarrow x = y \checkmark$.
by HW: could combine (i) and (iii)
by observation:
(iv) $x < y \implies y \not< x$

2) $\preceq$ is a strict partial order on $P(A)$ for any set $A$.
   \[ \forall x, y, z \in P(A) \wedge x \preceq y \preceq z \implies x \preceq z \]
   \[ x \not< y \implies \neg (y \not< x) \]

\[
\text{then empty.}
\]
1) $\preceq$ not a strict partial order on $R$ since it is not irreflexive
   (in fact $\preceq$ reflexive: this is stronger than being irreflexive!)
   Simply $y \preceq y$ is not a strict partial order on any set.

2) $\preceq$ are not (nonstrict) partial orders: neither are reflexive.

3) $\not\preceq$ (e.g. $\text{on } N$) is neither a partial order nor strict partial order since transitivity fails;
   e.g. $2 \not\preceq 5$ and $5 \not\preceq 2$ but $2 \preceq 2$. 

Total orders

**Defn** A relation $R$ on $A$ is said to be total if

$$\forall x, y \in A \ (x, y \in R \lor (y, x) \in R \lor x = y)$$

**Defn** If $R$ is a partial order on $A$ that is also total, then $R \cup -$ is called a total order on $A$.

If $R$ is a strict partial order on $A$ that is also total, then $R$ is called a strict total order on $A$.

**Ex's**

1. $\leq$ is a total order on $\mathbb{R}$; we knew already that $\leq$ is a partial order and

$$\forall x, y \in \mathbb{R} \ x \leq y \lor y \leq x \lor x = y$$

2. $\subseteq$ is not a total order on $P(\mathbb{N})$.

E.g. $E$ if $X = \{1, 2, 3\}$

$Y = \{3, 4\}$

then $X \not\subseteq Y$

and $X \not\supseteq Y$

3. $<$ is a strict total order on $\mathbb{R}$

Since $\forall x, y \in \mathbb{R} \ x < y \lor y < x \lor x = y$
Ex

A strict partial order on $\mathbb{N} \times \mathbb{N}$:

Define a relation $R$ on $\mathbb{N} \times \mathbb{N}$ by:

$$(n_1, m_1) R (n_2, m_2) \iff n_1 < n_2 \text{ and } m_1 < m_2$$

So e.g. $(1,2) R (3,5)$ since $1 < 3$ and $2 < 5$.

But $(3,1) \not R (2,2)$ since $3 \neq 2$.

Claim: $R$ is a strict partial order on $\mathbb{N} \times \mathbb{N}$

PF: we prove (1) transitivity (2) asymmetry

(1) Fix $(n_1, m_1), (n_2, m_2), (n_3, m_3) \in \mathbb{N} \times \mathbb{N}$ and suppose $(n_1, m_1) R (n_2, m_2)$ and $(n_2, m_2) R (n_3, m_3)$.

Then: $n_1 < n_2$ and $m_1 < m_2$ and $n_2 < n_3$ and $m_2 < m_3$.

Hence by transitivity $(n_1, m_1) \not R (n_3, m_3)$.

(2) Asymmetry: Assume $(n, m) R (n', m')$.

This implies $n < n'$ and $m < m'$.

Then $(n', m') \not R (n, m)$ since $n' \neq n$ and $m' \neq m$.

Hence $R$ is a strict partial order on $\mathbb{N} \times \mathbb{N}$. 
(vi)

\[ h_1 < h_3 \quad \text{and} \quad m_1 < m_3 \]

hence \( (h_1, m_1) \) \( R \) \( (h_3, m_3) \) \( \checkmark \)

(2) Fix \( (h_1, m_1), (h_2, m_2) \in \mathbb{N} \times \mathbb{N} \)

Suppose \( (h_1, m_1) \) \( R \) \( (h_2, m_2) \)

then \( h_1 < h_2 \) and \( m_1 < m_2 \)

hence \( h_2 \neq h_1 \) (and \( m_2 \neq m_1 \))

hence \( (h_2, m_2) \) \( \checkmark \) \( (h_1, m_1) \)

Observe: \( R \) defined above is \( \text{reflexive} \)

\( \text{total: e.g.} \)

\[ (1, 2) \ R \ (3, 1) \]

and \( (3, 1) \ R \ (1, 2) \)

and \( (5, 2) \neq (3, 1) \)