Equivalence

Given $P \iff Q$, the statement $P \iff Q$
(read: "P if and only if Q"
"P iff Q")

True if and only if $P, Q$ have
same truth value.

<table>
<thead>
<tr>
<th>$P$</th>
<th>$Q$</th>
<th>$P \iff Q$</th>
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1. $1+1 = 2 \iff 2+2 = 4 \quad \text{u (T)}$
2. $1+1 = 2 \iff 2+2 = 5 \quad \text{u (T)}$
3. $1+1 = 2 \iff 2+2 = 5 \quad \text{u (F)}$
4. $(\forall x \in \mathbb{R})(x > 0 \iff (\exists y \in \mathbb{R})(x = y^2)) \quad \text{u (T)}$

5. Why, for every $x \in \mathbb{R}$
the statements "$x > 0$"
and "$(\exists y \in \mathbb{R})(x = y^2)$"
are either
both true
or both false

Def’n: Statements $P, Q$ are logically equivalent

If $P \iff Q \text{ u true}$

E.g. $1+1 = 2$ and $2+2 = 4$ are logic equiv.
I am more interested in logically equivalent forms for connected statements, esp. negated statements.

Negation of Quantified Statements

1. $\neg (\forall x \in S) P(x)$
2. $\neg (\exists x \in S) P(x)$

Observe:

1. True if there is no $x \in S$ such that $P(x)$ is false, i.e., if $\neg \exists x \in S \neg P(x)$ is true.
2. True if for every $x \in S$, $P(x)$ is true, i.e., if $\forall x \in S \neg P(x)$ is true.

More succinctly, we have that

$$\neg (\forall x \in S) P(x) \iff (\exists x \in S) \neg P(x)$$

$\neg (\forall x \in S) P(x)$ is true (i.e., $\neg (\forall x \in S) P(x)$ and $(\exists x \in S) \neg P(x)$ are logically equivalent) and similarly

$$\neg (\exists x \in S) P(x) \iff (\forall x \in S) \neg P(x)$$

is true.
Ex's 1. \( \neg (\forall x \in \mathbb{R}) (x \in \mathbb{N}) \) "not all reals are naturals"
   is equiv to
   \( (\exists x \in \mathbb{R}) \neg (x \in \mathbb{N}) \) "there is a real which is not a natural."

   We can write \( \neg (x \in \mathbb{N}) \) as \( x \notin \mathbb{N} \).
   Similarly, we can write \( \neg (x = y) \) as \( x \neq y \).

2. \( \neg (\exists x \in \mathbb{R}) (x \in \mathbb{N}) \iff (\forall x \in \mathbb{R}) (x \notin \mathbb{N}) \)
   \( \neg \) true \( \iff \) because both inner statements are false.

   "There is no real which is a natural.

3. For multiple quantifiers; just iterate process.

   \( \neg (\forall x \in \mathbb{R}) (\exists y \in \mathbb{R}) (xy = 1) \) \( \iff (\exists x \in \mathbb{R}) (\forall y \in \mathbb{R}) \neg (xy = 1) \)

   "There is no real which is a multiplicative inverse."

   \( \exists (\forall x \in \mathbb{R}) (\forall y \in \mathbb{R}) (xy \neq 1) \)

   "There is a real that has no (inverse)."

   These are all true since \( x \neq 0 \) has no inverse.
Negating connected statements

**Theorem.** For any statements $P$, $Q$ the following equivalences hold:

1. $\neg \neg P \iff \neg P$
2. $\neg (P \land Q) \iff \neg P \lor \neg Q$
3. $\neg (P \lor Q) \iff \neg P \land \neg Q$.

To prove these equivalences, use truth tables:

**PF.**

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3. Similar
and \( \neg \) are called De Morgan's Laws for logic.

Ex's

1. \( \neg(1+1=2) \)
   is equiv. to
   \( 1+1 \neq 2 \) (both true)

2. \( \neg(1+1=2 \land 1+1=3) \)
   is equiv. to
   \( 1+1 \neq 2 \lor 1+1 \neq 3 \) (both true)

3. \( \neg(1+1=2 \lor 1+1=3) \)
   is equiv. to
   \( 1+1 \neq 2 \land 1+1 \neq 3 \) (false)

4. \( \forall x \in \mathbb{R}, \neg(x \leq 0 \lor \exists y \in \mathbb{R}) (y^2 = x) \)
   \( \iff \) \( \forall x \in \mathbb{R}, [\neg(x \leq 0) \lor \neg(\exists y \in \mathbb{R}) (y^2 = x)] \)
   \( \iff \) \( \forall x \in \mathbb{R}, [(x > 0) \lor \forall y \in \mathbb{R} \neg(y^2 = x)] \)
   (true)
Other useful logical equivalences

Then: the following equivalencies hold:

1. $P \Rightarrow Q \equiv (\neg P \lor Q)$
2. $(P \Rightarrow Q) \equiv (\neg Q \Rightarrow \neg P)$
3. $(P \iff Q) \equiv (P \Rightarrow Q \land Q \Rightarrow P)$

Proof: 1 and 2:

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<tr>
<th>P</th>
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<th>\neg P \lor Q</th>
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\[ (P \Rightarrow Q) \equiv (\neg P \lor Q) \quad (P \iff Q) \equiv (\neg Q \Rightarrow \neg P) \]

Try 3.

Then: the following equivalencies hold:

1. $\neg (P \Rightarrow Q) \equiv (P \land \neg Q)$
2. $\neg (P \iff Q) \equiv [\neg (P \land \neg Q) \lor \neg (\neg P \lor Q)]$
Pf: Instead of a table, we can use our previous equivalences.

1. \( \neg (P \Rightarrow Q) \iff \neg (P \lor Q) \)
   \( \iff \neg P \land \neg Q \)
   \( \iff P \land Q \quad \checkmark \)

2. \( \neg (P \Rightarrow Q) \iff \neg [P \Rightarrow (Q \land (Q \Rightarrow P))] \)
   \( \iff \neg [(P \lor Q) \land (Q \Rightarrow P)] \)
   \( \iff \neg (P \lor Q) \lor \neg (Q \Rightarrow P) \)
   \( \iff \neg (P \lor Q) \lor (Q \land P) \)

Ex's: Let \( E, O, P \) denote the sets of even, odd, and prime positive integers, resp.

1. \( S \cap O \Rightarrow 6 \in E \)
   \( \iff \neg \neg (S \cap O) \lor 6 \in E \)
   \( \iff (S \cap O) \lor 6 \in E \)
   Which we can write
   \( S \not\subseteq O \lor 6 \in E \quad (\text{True}) \)

2. \( \forall x \epsilon N \, (x \in E \Rightarrow x+1 \in E) \)
   \( \iff \forall x \epsilon N \, (x \not\in O \lor x+1 \in E) \)
   \( \iff \forall x \epsilon N \, (x \not\in O \lor x \not\in O \lor x \not\in O) \)
   \( \iff \forall x \epsilon N \, (x \not\in O \Rightarrow x \not\in O) \quad (\text{True}) \)
3. \((\forall x \in \mathbb{N})(x \in P \iff x \in \phi)\)
   is equivalent to
   \((\forall x \in \mathbb{N})[(x \in P \iff x \in \phi) \land (x \in \phi \iff x \in P)]\)
   (False)

4. Consider the true statement
   \((\forall x \in \mathbb{R}) [(x \geq 0) \iff (\exists y \in \mathbb{R})(x = y^2)]\)

Let \(\phi\) and its logical negation in positiva form:
\[\neg(\forall x \in \mathbb{R}) [(x \geq 0) \iff (\exists y \in \mathbb{R})(x = y^2)]\]
\[\exists x \in \mathbb{R} [(x \geq 0) \land \neg (\exists y \in \mathbb{R})(x = y^2)]\]
\[\neg (\exists x \in \mathbb{R}) [(x \geq 0) \land (\exists y \in \mathbb{R})(x = y^2)]\]
\[\neg (\exists x \in \mathbb{R})(x \geq 0) \lor \neg (\exists y \in \mathbb{R})(x = y^2)\]

Define a statement \(P\) is in positive form if any negation symbol in \(\overline{P}\) occur next to sub-statements that contain no connectives or quantifiers.

Only above allows you to find \(\overline{P}\) any \(P\) algebraically equivalent to \(P\) in positive form.
Thm (Associative + Distributive laws)
The following equivalences hold:

1. \((P \land Q) \land R \Leftrightarrow P \land (Q \land R)\)
2. \((P \lor Q) \lor R \Leftrightarrow P \lor (Q \lor R)\)
3. \(P \land (Q \lor R) \Leftrightarrow (P \land Q) \lor (P \land R)\)
4. \(P \lor (Q \land R) \Leftrightarrow (P \lor Q) \land (P \lor R)\)

For proofs see 4.6.3 and 4.6.4 in textbook.

Proving equality of sets using \(\Leftrightarrow\)’s

—there is a strong analogy between logical connectives and set operations introduced in ch. 5

<table>
<thead>
<tr>
<th>Connective</th>
<th>Operation</th>
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<tbody>
<tr>
<td>(P \land Q)</td>
<td>(A \land B)</td>
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<tr>
<td>(P \lor Q)</td>
<td>(A \lor B)</td>
</tr>
<tr>
<td>(P \Rightarrow Q)</td>
<td>(A \implies B)</td>
</tr>
<tr>
<td>(P \Leftrightarrow Q)</td>
<td>(A \iff B)</td>
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<tr>
<td>(\neg P)</td>
<td>(\neg A)</td>
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—analogy gives us a new way of proving equality of two sets, by a string of \(\Leftrightarrow\)’s.
Theorem \[ \text{Suppose } A, B \text{ are sets and } U \text{ is a universal set with } A, B \subseteq U. \]

Then we have:

1. \[ \overline{\overline{A}} = A \]
2. \[ A \cap B = \overline{A} \cup \overline{B} \]
3. \[ A \cup B = \overline{\overline{A} \cap \overline{B}} \]

Proof:

1. Fix \( x \in U \)

Then \( x \in A \iff x \notin \overline{A} \)

\( \iff \neg (x \in \overline{A}) \)

\( \iff \neg (x \notin A) \)

\( \iff x \in A \)

Thus the chain of equivalences shows \( x \in A \iff x \in \overline{A} \)

\[ x \in A \iff x \in \overline{A} \]

and \( x \in A \iff x \in \overline{A} \)

Here we've proved \( \overline{\overline{A}} = A \).

2. Fix \( x \in U \)

Then \( x \in A \overline{B} \iff x \notin A \cup B \)

\( \iff \neg (x \in A \cup B) \)

\( \iff \neg (x \in A \lor x \in B) \)

\( \iff x \in \overline{A} \land \overline{B} \)

\( \iff x \in \overline{A} \land \overline{B} \)

\( \iff x \in \overline{A \overline{B}} \)
Exercise: Use the distributive law \( P(A \cap (Q \cup R)) \iff (P \cap Q) \cup (P \cap R) \) to prove:

Theorem: For any sets \( A, B, C \) we have

\[ A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \]
Proof writing

Two approaches: when trying to prove a statement \( P \), can either prove \( P \) directly, or assume \( \neg P \) and derive a contradiction (i.e. prove \( \neg \neg P \)).

More generally: can prove any statement logically equiv. to \( P \), or disprove any statement logically equiv. to \( \neg P \).

Existence Proofs

General form (\( \exists x \in S \)) \( P(x) \)

Direct proof strategy: define an \( x \) and prove \( P(x) \) holds.

Ex 1 Prop'n: There is an even number that can be written as the sum of two primes in two distinct ways.

\( 24 = 17 + 7 = 19 + 5 \)

PF: Consider \( n = 10 \).
- Then \( n \) is even and we have \( n = 5 + 5 \) and \( n = 7 + 3 \).
- Since 3, 5, 7 are primes, the prop'n is proved.

Note: \( n = 24 = 19 + 5 = 17 + 7 \) works too...
Indirect Proof Strategy:
- Assume \( \neg (\exists x \in S) P(x) \)
  and derive a contradiction
- Equivalently, assume \((\forall x \in S) \neg P(x)\)
  and get a contradiction

Ex. 2 Fix \( n \in \mathbb{N} \) and suppose \( a_1, \ldots, a_n \).

Then at least one of \( a_1, \ldots, a_n \)
are at least as large as their average.
That is:
\[
(\exists k \in [n]) \left( a_k \geq \frac{1}{n} \left( a_1 + \ldots + a_n \right) \right)
\]
\[
\frac{1}{n} \sum_{i=1}^{n} a_i
\]

PF. Suppose not, toward a contradiction.
That is: Suppose that
\[
(\forall k \in [n]) (a_k < \frac{1}{n} (a_1 + \ldots + a_n))
\]
- For simplicity let \( S = a_1 + \ldots + a_n \).
- Our assumption is, for every \( k \in [n] \) we have
  \( a_k < S/n \).

But then we have
\[
S = a_1 + a_2 + \ldots + a_n < \frac{S}{n} + \frac{S}{n} + \ldots + \frac{S}{n}
\]
by our assumption.
\[ n \left( \frac{5}{n} \right) = 5 \]

- This shows \( s < s \), a contradiction.
- Thus our assumption was false, and hence the property must be true.

Universal Proofs

General Form: \((\forall x \in S) P(x)\)

Direct Strategy:
- Let \( x \in S \) be arbitrary but fixed.
- Prove \( P(x) \) holds.

\textbf{Ex 1: Propn:} \((\forall x, y \in \mathbb{R}) (xy \leq (x^2 + y^2)/2)\)

\textbf{PF:}
- Fix \( x, y \in \mathbb{R} \).
- Since squares are always non-negative, we have \( 0 \leq (x - y)^2 \).
- Hence \( 0 \leq x^2 - 2xy + y^2 \).
- Hence \( 2xy \leq x^2 + y^2 \).
- \( \therefore \) \( xy \leq (x^2 + y^2)/2 \).

- Since \( x, y \) were arbitrary, the propn is proved.
Note: Propn is one version of "AMGM" inequality

\[ \Rightarrow \text{ arithmetic mean of } x, y = \frac{x + y}{2} \text{ (AM)} \]

\[ \Rightarrow \text{ geometric mean of } x, y = \sqrt{xy} \text{ (GM)} \]

Propn gives for \( x, y \geq 0 \)

\[ \sqrt{xy} \leq \frac{x + y}{2} \]

i.e. \[ \text{GM} \leq \text{AM}. \]

Indirect proof:

- Assume \( \forall x \in S \) \( P(x) \)

(i.e. \( \exists x \in S \) \( \neg P(x) \))

and get a contradiction

Ex 2: \( P(x) \): \( \sqrt{2} \) is irrational,

then \( \frac{a}{b} \neq \sqrt{2} \)

\( \forall a, b \in \mathbb{Z} \) (\( \frac{a}{b} \neq \sqrt{2} \))

PF: Suppose \( \frac{a}{b} = \sqrt{2} \), that is, suppose \( \exists a, b \in \mathbb{Z} \) s.t.

\[ \frac{a}{b} = \sqrt{2} \]

- we may assume \( \frac{a}{b} \) is in reduced form i.e. \( a \) and \( b \)

have no common factors since if they do we can cancel and get a fraction in reduced form.