Homework #9

1. Let \( F \) denote the set of all functions from \( \mathbb{N} \) to \( \mathbb{N} \), that is, \( F = \{ f \subseteq \mathbb{N} \times \mathbb{N} \mid f \) is a function\}. Define a relation \( R \) on \( F \) by the rule \( (f, g) \in R \) iff for every \( n \in \mathbb{N} \) we have \( f(n) \leq g(n) \). Prove that \( R \) is a partial order on \( F \).

2. Fix \( m, n \in \mathbb{N} \). Define a mapping \( f : \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z} \) by \( f([a]_n) = [a]_m \).
   a. Prove that if \( m \mid n \) then \( f \) is a well-defined function. That is, prove that if \( [a]_n = [b]_n \) then \( f([a]_n) = f([b]_n) \).
   b. Let \( n = 12 \) and \( m = 3 \). Write \( \text{PreIm}_f([1, 2]) \) in roster notation.
   c. Suppose \( m \nmid n \). Show that \( f \) is ill-defined. That is, show there exist \( a, b \in \mathbb{Z} \) such that \( [a]_n = [b]_n \) but \( f([a]_n) \neq f([b]_n) \).

3. Suppose that \( A, B, \) and \( C \) are nonempty sets and \( f : A \to B \) and \( g : B \to C \) are functions.
   a. Prove that if \( f \) and \( g \) are surjections then so is \( g \circ f \).
   b. Prove that if \( f \) and \( g \) are injections then so is \( g \circ f \).
   c. Use your results from parts (a.) and (b.) to prove that if \( f \) and \( g \) are bijections then so is \( g \circ f \).

4. Suppose \( X \) and \( Y \) are nonempty sets and \( f : X \to Y \) is a function. Define a new function \( F : \mathcal{P}(Y) \to \mathcal{P}(X) \) by \( F(B) = \text{PreIm}_f(B) \). Prove that \( F \) is injective if and only if \( f \) is surjective.