Homework #4

1. Consider the following variable propositions:

Let \( P(x) \) be the proposition “\( 1 \leq x \leq 3 \)”
Let \( Q(x) \) be the proposition “\( (\exists k \in \mathbb{Z})(x = 2k) \)”
Let \( R(x) \) be the proposition “\( x^2 = 4 \)”

Recall that a statement is in positive form if the only negation symbols in the statement appear next to substatements that do not contain quantifiers or connectives.

For each of the following statements, write the negation in a logically equivalent positive form. Then decide which claim (the original or the negation) is true (no proof required).

a.) \( (\forall x \in \mathbb{Z})(P(x) \Rightarrow Q(x)) \)

b.) \( (\exists x \in \mathbb{Z})(R(x) \land P(x)) \)

c.) \( (\forall x \in \mathbb{Z})(R(x) \Rightarrow P(x)) \)

d.) \( (\forall x \in \mathbb{Z})(\exists y \in \mathbb{Z})(x \neq y \land P(x) \land Q(x)) \)

2. Consider the following propositions, which assert that the rational numbers are dense, and the integers are discrete, respectively:

(a) Strictly between any two distinct rational numbers lies a third rational number.

(b) For every integer \( n \), there is a strictly larger integer \( m \), such that there are no integers strictly between \( n \) and \( m \).

Write out these propositions symbolically, using only logical symbols and the sets \( \mathbb{Q} \) and \( \mathbb{Z} \).

3. For every \( i \in \mathbb{N} \), define a set \( A_i \subseteq \mathbb{N} \) such that the indexed family of sets \( \{A_i : i \in \mathbb{N}\} \) satisfies all of the following properties (recall that \( \subseteq \) means “is a strict subset of”):

a.) \( (\forall n \in \mathbb{N})(\exists i \in \mathbb{N})(n \in A_i) \)

b.) \( (\forall i \in \mathbb{N})(\exists n \in \mathbb{N})(n \notin A_i) \)

c.) \( (\forall i, m \in \mathbb{N})(\exists n \in \mathbb{N})(n > m \land n \in A_i) \)

d.) \( (\exists j \in \mathbb{N})(\forall i \in \mathbb{N})(i \neq j \Rightarrow A_j \subsetneq A_i) \)

Then, prove that the family you’ve defined satisfies each of these properties.

4. Use a chain of logical equivalences to prove the following propositions.

a.) Given a universal set \( U \) and sets \( A, B \subseteq U \), it is the case that \( (A \cup B) \cap \overline{A} = B - A \).

b.) For all sets \( A, B, \) and \( C \), it is the case that \( A \cap (B - C) = (A \cap B) - (A \cap C) \).

(A possibly helpful hint: if \( P \) is a false statement, then \( P \lor Q \) is logically equivalent to \( Q \).)

5. Consider the following proposition:

For all integers \( n \), \( n \) is an integer multiple of 3 if and only if \( n^2 - 1 \) is not a multiple of 3.

a.) Write out this proposition symbolically, using only logical symbols and the set \( \mathbb{Z} \).

b.) Prove the proposition. (You should be able to prove it using nothing more than the definition of being a multiple of 3, and the fact that every integer has a remainder of 0, 1, or 2 when divided by 3.)