Homework #2

1. Let \((a, b) \in \mathbb{R}^2\) and fix \(\epsilon \in \mathbb{R}\) with \(\epsilon > 0\). Define \(C_{(a,b),\epsilon}\) as the set of real numbers “within \(\epsilon\)” of \((a, b)\):

\[
C_{(a,b),\epsilon} = \{(x, y) \in \mathbb{R}^2 | \sqrt{(x-a)^2 + (y-b)^2} < \epsilon\}.
\]

a.) Give a geometric description of \(C_{(a,b),\epsilon}\).

b.) Identify the following sets. Write your answer in the form of \(C_{(a,b),\epsilon}\) or as one of the standard sets discussed in class.
   i. \(C_{(0,0),1} \cap C_{(0,0),2}\)
   ii. \(C_{(0,0),1} \cup C_{(0,0),2}\)
   iii. \(C_{(0,0),1} \cap C_{(2,2),1}\)

c.) For a given \(\epsilon > 0\), define \(D_{(a,b),\epsilon}\) as follows:

\[
D_{(a,b),\epsilon} = \{(x, y) \in \mathbb{R}^2 | \sqrt{(x-a)^2 + (y-b)^2} \leq \epsilon\}.
\]

What is \(D_{(a,b),\epsilon} - C_{(a,b),\epsilon}\) geometrically? Write a definition for this set using set-builder notation.

2. Let \(A, B,\) and \(C\) be sets. Prove that

\[
A - (B - C) \subseteq (A - B) \cup C
\]

and then provide an example of sets \(A, B,\) and \(C\) for which the containment is strict.

3. Let \(A\) and \(B\) be sets, and suppose that \(\mathcal{P}(A) = \mathcal{P}(B)\). Is it necessarily the case that \(A = B\)? If so, prove it. If not, provide a counterexample.

4. For each \(n \in \mathbb{N}\), let \(A_n = [n] \times [n]\). Define \(B = \bigcup_{n \in \mathbb{N}} A_n\). Does \(B = \mathbb{N} \times \mathbb{N}\)? Either prove that it does, or show why it does not.

5. Let \(I = \{x \in \mathbb{R} | 0 < x < 1\}\). For each \(x \in I\), define \(S_x = \{y \in \mathbb{R} | x < y < x + 1\}\). Provide a double containment proof that

\[
\bigcup_{x \in I} S_x = \{z \in \mathbb{R} | 0 < z < 2\}.
\]