10.3 Polar Coordinates

- A point \( P \) in the plane is uniquely specified by its rectangular coordinates \((x, y)\).

- Can also specify \( P \) by its polar coordinates \((r, \theta)\), where
  - \( r \) = distance to origin
  - \( \theta \) = angle made w/ \( x \)-axis

**Example:**
\[ P = (2, \pi/4) \]
\[ Q = (2, 5\pi/4) \] are shown below.
We allow $\theta > 2\pi$ and $\theta < 0$, e.g., $P$ also has coords $(2, \frac{\pi}{4})$ and $(2, -\frac{\pi}{4})$

So, polar coords are not unique!
Also allow $r < 0$, e.g., $Q = (-2, \frac{\pi}{4})$
To translate:

from polar to rect: use: \[ x = r \cos \theta \]
\[ y = r \sin \theta \]

from rect. to polar, use: \[ r^2 = x^2 + y^2 \]
\[ \tan \theta = \frac{y}{x} \]

ex: if P has polar coords \((2, \frac{\pi}{4})\)
then P has rectangular coords
\[ x = 2 \cos \left( \frac{\pi}{4} \right) = 2 \cos \left( \frac{\pi}{4} \right) = \sqrt{2} \]
\[ y = 2 \sin \left( \frac{\pi}{4} \right) = 2 \sin \left( \frac{\pi}{4} \right) = \sqrt{2} \]
If $P$ has rectangular co-ords $(3, 4)$ then $P$ has polar co-ords given by

$$r^2 = 3^2 + 4^2 = 25 \Rightarrow r = 5$$

$$\theta = \tan^{-1} \left( \frac{4}{3} \right) = 0.927\ldots$$

**Polar curves**

- can also specify curves w/ polar equations
- usually we consider eq'n of the form $r = f(\theta)$, i.e. when $r$ is a function of $\theta$.
- graph is all polar points $(r, \theta)$ when $r = f(\theta)$.

**Ex:** graph the polar curve $r = 2$. \\
**Soln:** consists of all points $(r, \theta)$ where $r = 2$. 
to graph more complicated curves $r = f(\theta)$, can take various approaches:

- plot points (not usually effective in itself)
- translate to rectangular coords (doesn't always work)
- use calculus to find points $\frac{1}{7}$
  - horiz. or vert. tan lines (see second ex below)
  - on brain.

ex: graph $r = \cos \theta = 2 \cos \theta$

Solt: can plot some points to start.
(vi) \[ r = 2 \cos \theta \]

<table>
<thead>
<tr>
<th>( \theta )</th>
<th>0</th>
<th>( \pi/4 )</th>
<th>( \pi/2 )</th>
<th>( 3\pi/4 )</th>
<th>( \pi )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( r )</td>
<td>2</td>
<td>( \sqrt{2} )</td>
<td>0</td>
<td>( -\sqrt{2} )</td>
<td>-2</td>
</tr>
</tbody>
</table>

- Only gives a very rough sense of curve.
- In this case we can translate to rectangular co-ords, but requires some creativity.

Use: \[ x = r \cos \theta \quad y = r \sin \theta \quad x^2 + y^2 = r^2 \]

To get \[ r = 2 \cos \theta \] into only \( x, y \).

\[ \Rightarrow \cos \theta = \frac{x}{r} \]

\[ \Rightarrow r = 2 \cdot \frac{x}{r} \]

\[ \Rightarrow r^2 = 2x \]

\[ \Rightarrow x^2 + y^2 = 2x \]

\[ \Rightarrow x^2 - 2x + y^2 = 0 \]

\[ \Rightarrow (x - 1)^2 + y^2 = 1 \]

Circle of radius 1 centered at \((1,0)\).
(vii) Using Calculus to graph a curve \( r = f(\theta) \), finding \( \frac{dy}{dx} \) at various points more reliably than plotting points randomly and trying to interpolate.

- So we need formula for \( \frac{dy}{dx} \)

- Using \( x = r \cos \theta = f(\theta) \cos(\theta) \)
  \[ y = r \sin \theta = f(\theta) \sin(\theta) \]

Can view a polar curve \( r = f(\theta) \) as a parametric curve (using \( \theta \) as parameter instead of \( t \))

- We have \( \frac{dx}{d\theta} = \frac{dr}{d\theta} \cos \theta = r \sin \theta \)
  \[ \frac{dy}{d\theta} = \frac{dr}{d\theta} \sin \theta + r \cos \theta \]

- From before we knew
  \[ \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\frac{dr}{d\theta} \sin \theta + r \cos \theta}{\frac{dr}{d\theta} \cos \theta - r \sin \theta} \]
(viii)

Ex: 10) For the curve \( r = 1 + \sin \theta \), find the points at which the line is horizontal or vertical, for \( 0 \leq \theta \leq 2\pi 

b) Sketch the curve for \( 0 \leq \theta \leq 2\pi \).

Sol'n: 1) We know 
\[
\frac{dy}{dx} = \frac{\frac{dr}{d\theta} \sin \theta + r \cos \theta}{\frac{dr}{d\theta} \cos \theta - r \sin \theta}
\]

2) horizontal: when \( \frac{dy}{dx} = 0 \)

\[
1 \cdot \frac{dr}{d\theta} \sin \theta + r \cos \theta = 0
\]

3) \( \frac{dr}{d\theta} \cos \theta - r \sin \theta = 0 \)

4) \( r = 1 + \sin \theta \) so \( \frac{dr}{d\theta} = \cos \theta \)

5) So we want: \( \cos \theta \sin \theta + (1 + \sin \theta) \cos \theta = 0 \)

6) \( \sin \theta = -1 \) or \( \sin \theta = 0 \)

7) \( \cos \theta = 0 \) or \( \sin \theta = -\frac{1}{2} \)

8) \( \cos \theta = 0 \) or \( \sin \theta = -\frac{1}{2} \)

9) \( \theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{7\pi}{6}, \frac{11\pi}{6} \)

At these \( \theta \)'s we have:

\[
\begin{align*}
\theta = \frac{\pi}{2} & : r \left( \frac{\pi}{2} \right) = 1 + \sin \left( \frac{\pi}{2} \right) = 2 \\
\theta = \frac{3\pi}{2} & : r \left( \frac{3\pi}{2} \right) = 1 + \sin \left( \frac{3\pi}{2} \right) = 0 \\
\theta = \frac{7\pi}{6} & : r \left( \frac{7\pi}{6} \right) = 1 + \sin \left( \frac{7\pi}{6} \right) = \frac{1}{2} \\
\theta = \frac{11\pi}{6} & : r \left( \frac{11\pi}{6} \right) = 1 + \sin \left( \frac{11\pi}{6} \right) = \frac{1}{2}
\end{align*}
\]
Solve for: \( \frac{dr}{d\theta} \cos \theta - r \sin \theta = 0 \)

\[ \cos^2 \theta - (1 + \sin \theta) \sin \theta = 0 \]
\[ \cos^2 \theta - \sin \theta - \sin^2 \theta = 0 \]
\[ 1 - \sin^2 \theta - \sin^2 \theta - \sin \theta = 0 \]
\[ 1 - \sin \theta - 2 \sin^2 \theta = 0 \]
\[ (1 - 2 \sin \theta)(1 + \sin \theta) = 0 \]

\[ \sin \theta = -1 \Rightarrow \frac{5\pi}{2} \]

or \( \sin \theta = \frac{1}{2} \Rightarrow \frac{\pi}{6}, \frac{5\pi}{6} \)

at these pts

\[ r(\frac{5\pi}{2}) = 1 + \sin(\frac{5\pi}{2}) = 0 \]
\[ r(\frac{\pi}{6}) = 1 + \frac{1}{2} = \frac{3}{2} = r(\frac{5\pi}{6}) \]

let's graph these points:
Observe: $E \left( \frac{\pi}{2}, \frac{7\pi}{6} \right), \left( \frac{\pi}{2}, \frac{11\pi}{6} \right), \left( 2, \frac{\pi}{2} \right)$

Horizontal line $\sin \frac{dy}{dx} = 0$ is zero

$E \left( \frac{3\pi}{2}, \frac{5\pi}{6} \right), \left( \frac{3\pi}{2}, \frac{\pi}{6} \right)$

Vertical line $\lim \sin \frac{dy}{dx} = \text{undefined}$

But $E \left( 0, \frac{5\pi}{6} \right)$ unclear

Since $\frac{dy}{dx} = \frac{0}{0}$.

We use L'Hopital:

$$\lim_{\theta \to \frac{3\pi}{2}} \frac{dy}{dx} = \lim_{\theta \to \frac{3\pi}{2}} \frac{\cos \theta (1 + 2 \sin \theta)}{(1 + \sin \theta)(1 - 2 \sin \theta)}$$

$$= \lim_{\theta \to \frac{3\pi}{2}} \frac{\cos \theta}{1 + \sin \theta} \lim_{\theta \to \frac{3\pi}{2}} \frac{1 + 2 \sin \theta}{1 - 2 \sin \theta}$$

$$= \frac{11}{6} \cdot \frac{-1}{3} = \frac{11}{18}$$
\[
\lim_{\theta \to \pi/2} \frac{\cos \theta}{1 + \sin \theta} = -\frac{1}{2} \lim_{\theta \to \pi/2} \frac{-\sin \theta}{\cos \theta} = -\frac{1}{2} \cdot (\infty) = -\infty.
\]

So there is a vertical line at this point.
Integration and Length in polar co-ords

Integration in rectangular co-ords:

To find area under $y = f(x)$ between $x = a$ and $x = b$

Approximate:

Segment area

$\approx f(a) \Delta x$

Total area

$\approx \sum f(x) \Delta x$

Then take limit:

area $= \int_a^b f(x) \, dx$

$= \int_a^b y \, dx$
In polar coordinates:

to find area bounded by
polar curve $r = f(\theta)$ between $\theta = \alpha$
and $\theta = \beta$

First:

approximate:

approx this area by
a circular sector

sector area
$$= \frac{1}{2} r^2 \Delta \theta$$

(Why: entire area of circle corresponds to $\theta = 2\pi$, which gives $\text{Area} = \frac{1}{2} r^2 2\pi = \pi r^2$)
So area bounded by $r = f(\theta)$

between $\theta = \alpha, \beta$ is approx:

$$
\sum 2 \frac{1}{2} r_i^2 (\Delta \theta)
= \sum 2 \frac{1}{2} (f(\theta))^2 \Delta \theta
$$

So exact area (taking limit) is:

$$
\int_{\alpha}^{\beta} \frac{1}{2} r^2 \, d\theta
= \int_{\alpha}^{\beta} \frac{1}{2} (f(\theta))^2 \, d\theta
$$

**EX:** Find the area enclosed by the cardioid $r = 1 + \sin \theta$

(i) in first quadrant

(ii) overall.
(iv) we knew $r = 1 + \sin\theta$ looks like.

\[ \theta = \pi/2 \text{ Per (i) we have this area} \]

\[ \theta = 0 \]

(i)

\[
A = \int_{0}^{\pi/2} \frac{1}{2} r^2 \, d\theta
\]

\[
= \frac{1}{2} \int_{0}^{\pi/2} (1 + \sin^2 \theta)^2 \, d\theta
\]

\[
= \frac{1}{2} \int_{0}^{\pi/2} 1 + 2\sin^2 \theta + \sin^2 \theta \, d\theta
\]

\[
= \frac{1}{2} \int_{0}^{\pi/2} 1 + 2\sin^2 \theta + \frac{1}{2} - \frac{1}{2} \cos 2\theta \, d\theta
\]

\[
= \frac{1}{2} \int \left[ e - 2\cos \theta + \frac{1}{2} e - \frac{1}{4} \sin 2\theta \right]_{0}^{\pi/2}
\]

\[
= \frac{1}{2} \left[ \left( \frac{3}{2} e - 2\cos \theta - \frac{1}{4} \sin 2\theta \right) \right]_{0}^{\pi/2}
\]

\[
= \frac{1}{2} \left[ \left( \frac{3}{2} e - 2\cos \frac{\pi}{2} - \frac{1}{4} \sin (2 \cdot \frac{\pi}{2}) \right) - \left( \frac{3}{2} e - 2\cos 0 - \frac{1}{4} \sin 0 \right) \right]
\]

\[
= \frac{1}{2} \left[ \left( \frac{3}{2} e - \frac{1}{4} \right) - \left( \frac{3}{2} e \right) \right]
\]

\[
= \frac{1}{2} \left( \frac{3}{2} e - \frac{1}{4} \right)
\]

\[
= \frac{1}{2} \left( \frac{3 \pi/4 + 2}{8} \right)
\]

\[
= \frac{1}{2} \left( \frac{3 \pi/4 + 2}{8} \right)
\]
(ii) Cycle thru entire cordial once over $0 \leq \theta \leq 2\pi$.

So

\[ \int_0^{2\pi} \frac{1}{2} (1 + \sin^2 \theta)^2 \, d\theta = \int_0^{2\pi} \frac{1}{2} \left[ \frac{3}{2} e - 2 \cos^2 \theta - \frac{1}{4} \sin 2 \theta \right] \, d\theta = \frac{1}{2} \left[ \left( \frac{3}{2} \cdot 2\pi \right) - 2 - 0 \right] - \left( 0 - 2 - 0 \right) = \frac{1}{2} \left[ 3\pi \right] = \frac{3\pi}{2} \checkmark

Example: Find area of region inside the circle $r = \sin \theta$ and outside the cardioid $r = 1 + \sin \theta$.

Solution: First: Why is $r = \sin \theta$ a circle?

Can translate to rectangular coordinates using $x = r \cos \theta$

$y = r \sin \theta$

$x^2 + y^2 = r^2$
(vi)

\[ \Rightarrow \sin \theta = \frac{y}{r} \]

So

\[ r = 3 \frac{y}{r} \]

\[ \Rightarrow r^2 = 3y \]

\[ \Rightarrow x^2 + y^2 = 3y \]

\[ \Rightarrow x^2 + y^2 - 3y = 0 \]

\[ \Rightarrow x^2 + y^2 - 3y + \left( \frac{3}{2} \right)^2 = \left( \frac{3}{2} \right)^2 \]

\[ \Rightarrow x^2 + (y - \frac{3}{2})^2 = \left( \frac{3}{2} \right)^2 \]

So, graph of \( r = 3 \sin \theta \) is:

\[ \text{corded peaks } C(0, \frac{3}{2}) \]
Interested in this area

Need to find angles of intersection.

Intersect when:

\[ 3 \sin \theta = 1 + \sin \theta \]

\[ \Rightarrow 2 \sin \theta = 1 \]

\[ \Rightarrow \sin \theta = \frac{1}{2} \]

\[ \Rightarrow \theta = \frac{\pi}{6}, \frac{5\pi}{6} \]

To find area between:

Subtract area of cardioid from area of circle over \( \frac{\pi}{6} \leq \theta \leq \frac{5\pi}{6} \).
(viii) 

\[ A = \int_{\pi/6}^{5\pi/6} \frac{1}{2} (3 \sin \theta)^2 \, d\theta - \int_{\pi/6}^{5\pi/6} \frac{1}{2} (1 + \sin \theta)^2 \, d\theta \]

\[ = \pi \]